Advanced Calculus Homework # 2 (due 10/16)

1. Let x_n be a monotone increasing sequence such that $x_{n+1} - x_n \leq 1/n$. Must x_n converge?

2. Let $\mathbb{F}$ be an ordered field in which every *strictly* monotone increasing sequence bounded above converges. Prove that $\mathbb{F}$ is complete.

3. Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ be bounded below and define $A + B = \{x + y | x \in A \text{ and } y \in B\}$. Is it true that $\inf(A + B) = \inf A + \inf B$?

4. Let $S \subset [0, 1]$ consist of all infinite decimal expansions $x = 0.a_1a_2a_3 \cdots$ where all but finitely many digits are 5 or 6. Find $\sup S$.

5. Let x_n be a sequence with $\limsup x_n = b \in \mathbb{R}$ and $\liminf x_n = a \in \mathbb{R}$. Show that x_n has a subsequences u_n and ℓ_n with $u_n \to b$ and $\ell_n \to a$.

6. Let A and B be two nonempty sets of real numbers with the property that $x \leq y$ for all $x \in A$ and $y \in B$. Show that there exists a number $c \in \mathbb{R}$ such that $x \leq c \leq y$ for all $x \in A$ and $y \in B$. Give a counterexample to this statement for rational numbers.

7. For nonempty sets $A, B \subset \mathbb{R}$, determine which of the following statements are true. Prove the true statements and give a counterexample for those that are false:

a. $\sup(A \cap B) \leq \inf\{\sup(A), \sup(B)\}$.
b. $\sup(A \cap B) = \inf\{\sup(A), \sup(B)\}$.
c. $\sup(A \cup B) \geq \sup\{\sup(A), \sup(B)\}$.
d. $\sup(A \cup B) = \sup\{\sup(A), \sup(B)\}$.

8. Let x_n be a sequence in $\mathbb{R}$ such that $d(x_n, x_{n+1}) \leq d(x_{n-1}, d_{x_n})/2$. Show that x_n is a Cauchy sequence.

9. Let $P \subset \mathbb{R}$ be a set such that $x \geq 0$ for all $x \in P$ and for each integer k there is an $x_k \in P$ such that $kx_k \leq 1$. Prove that $0 = \inf(P)$.

10. Assume that $A = \{a_{m,n} |\ m = 1, 2, 3, \cdots \text{ and } n = 1, 2, 3, \cdots\}$ is a bounded set and that $a_{m,n} \geq a_{p,q}$ whenever $m \geq p$ and $n \geq q$. Show that

$$\lim_{n\to\infty} a_{n,n} = \sup A.$$