高微第一次期中考解答

#6. No. no matter $n>1$ or $n=1$.

For example, let $V = V_1 \cup V_2$ where $V_1 = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_1 > 0\}$

$V_2 = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_1 < -1\}$. V is open in $\mathbb{R}^n$.

Define $f : V \longrightarrow \mathbb{R}^n$ by

$$f(x) = f(x_1, \cdots, x_n) = \begin{cases} x & \text{for } x \in V_1 \\ x + (3, 0, \cdots, 0) & \text{for } x \in V_2 \end{cases}$$

Then $f \in C'(V)$ and $Df(x) = I_n$ which is invertible $\forall x \in V$.

But $f(1, 0, \cdots, 0) = (1, 0, \cdots, 0) = f(-2, 0, \cdots, 0)$

i.e. f is not 1-1 on V.

#7. (i). Yes. By Thm 10.46.

(ii). No. By Rmk 10.47.

(iii). Yes. By Heine-Borel Thm, compact $\Leftrightarrow$ bdd & closed.

By §10.1 Ex 10.(a), sequentially compact $\Rightarrow$ bdd. & closed.

By §10.4 Ex 10.(a), compact $\Rightarrow$ sequentially compact.

(iv). No. Let ρ be the discrete metric in $\mathbb{R}^n$, then ρ is bounded.

(v). No. See Example 11.11.