§7.4

#5. (a) By Thm 7.44 and $\frac{d^n}{dx^n}e^x = e^x$ $\forall n \in \mathbb{N}$, for each $x \in [0,1]$

$$e^x - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) = \frac{e^c}{4!}x^4$$

for some $c \in [0, x]$

$$0 \leq \frac{e^c}{4!}x^4 \leq \frac{e}{4!} < \frac{3}{4!} = \frac{1}{8}$$

$$\therefore\ 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \leq e^x < 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{1}{8}$$
$$= \frac{9}{8} + x + \frac{x^2}{2} + \frac{x^3}{6}$$

#5(b). $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$ $\qquad$ $\frac{d^2}{dx^2}\tan^{-1}x = \frac{-2x}{(1+x^2)^2}$

$\frac{d^3}{dx^3}\tan^{-1}x = \frac{-2+6x^2}{(1+x^2)^3}$ $\qquad$ $\frac{d^4}{dx^4}\tan^{-1}x = \frac{-24x(x^2-1)}{(1+x^2)^4}$.

Thm 7.44 $\Rightarrow$ for each $x \in [0,1]$

$$\tan^{-1}x - \left(x - \frac{x^3}{3}\right) = \frac{-24c(c^2-1)}{(1+c^2)^4}\cdot\frac{x^4}{4!}$$
$$= \frac{c(1-c^2)}{(1+c^2)^4}x^4$$

for some $c \in [0, x] \subset [0,1]$.

$$0 \leq \frac{c(1-c^2)}{(1+c^2)^4}x^4 \leq \frac{c}{(1+c^2)^4} \leq \frac{c}{1+4c^2} \leq \frac{c}{4c} = \frac{1}{4}$$

Bernoulli's ineq. $\qquad$ A.M. $\geq$ G.M.

$$\therefore\ x - \frac{x^3}{3} \leq \tan^{-1}x \leq \frac{1}{4} + x - \frac{x^3}{3}.$$

#5(c). $\frac{d\log x}{dx} = \frac{1}{x}$, $\frac{d^2}{dx^2}\log x = -\frac{1}{x^2}$, $\frac{d^3}{dx^3}\log x = \frac{2}{x^3}$

$\frac{d^4}{dx}\log x = \frac{-6}{x^4}$.

<u>Thm 7.44</u> $\Rightarrow$ for each $x \in [1,2]$ $\quad (y = x-1 \in [0,1])$

$$\log x - \left(y - \frac{y^2}{2} + \frac{y^3}{3}\right) = -\frac{y^4}{4c^4}$$

for some $c \in [1, x] \subset [1,2]$

$$1 \le c \le 2$$

$$\Rightarrow \quad 1 \le c^4 \le 16$$

$$\Rightarrow \quad -\frac{y^4}{4} \le \frac{-y^4}{4c^4} \le \frac{-y^4}{64}.$$

$$\therefore \; y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} \le \log x \le y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{64}.$$

#6(a). $\frac{d\sin x}{dx} = \cos x$, $\frac{d^2}{dx^2}\sin x = -\sin x$, $\frac{d^3}{dx^3}\sin x = -\cos x$

For $0 < \delta \le 1$,

$$\sin\delta - \delta = -\frac{\cos c}{3!}\delta^3 \quad \text{for some } c \in [0, \delta]$$

$$|\sin\delta - \delta| = \left|\frac{\cos c}{3!}\delta^3\right| \le \frac{\delta^3}{3!}$$

$$\|$$

$$|\delta - \sin\delta| = |\delta + \sin(\delta + \pi)|$$

<u>(b)</u>. By (a), if $|x - \pi| \le \delta \le 1$,

<u>Case $0 \le x - \pi \le \delta$</u>

$$|x - \pi + \sin(x - \pi + \pi)| = |x - \pi + \sin x| \le \frac{(x-\pi)^3}{3!} \le \frac{\delta^3}{3!}$$

<u>Case $0 \le \pi - x \le \delta$</u>

$$|\pi - x + \sin(\pi - x + \pi)| = |\pi - x - \sin x| = |x - \pi + \sin x|$$

$$\le \frac{(\pi - x)^3}{3!} \le \frac{\delta^3}{3!}.$$

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#7. For each $x \in \mathbb{R}$, by Thm 7.52,

$$f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)}(t)\, dt$$

$$= \frac{1}{n!}\int_0^x s^n f^{(n+1)}(x-s)\, ds$$

$$\to 0 \quad \text{as} \quad n \to \infty$$

$\therefore$ $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ for each $x \in \mathbb{R}$.

By Thm 7.48, f is analytic on $\mathbb{R}$.

#9. ($\Rightarrow$) Given $x_0 \in (a,b)$, $\exists$ $c, d \in (a,b)$, $c < x_0 < d$ s.t.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k \quad \forall x \in (c,d)$$

As in the proof of Thm 7.39, by Thm 7.30,

$$f'(x) = \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{(k-1)!} (x-x_0)^{k-1} \quad \forall x \in (c,d).$$

$\therefore$ f' is analytic on (a,b).

($\Leftarrow$) Given $x_0 \in (a,b)$, $\exists$ $c, d \in (a,b)$, $c < x_0 < d$ s.t.

$$f'(x) = \sum_{k=0}^{\infty} \frac{f^{(k+1)}(x_0)}{k!} (x-x_0)^k \quad \forall x \in (c,d).$$

By Thm 7.32, for each $x \in (c,d)$,

$$\int_{x_0}^{x} f'(x)\, dx = \sum_{k=0}^{\infty} \frac{f^{(k+1)}(x_0)}{k!} \int_{x_0}^{x} (t-x_0)^k\, dt$$

$$\begin{array}{ccc} \| & & \| \\ f(x) - f(x_0) & & \sum_{k=0}^{\infty} \frac{f^{(k+1)}(x_0)}{(k+1)!} (x-x_0)^{k+1} \end{array}$$

$$= \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$

$\therefore$ $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k \quad \forall x \in (c,d)$

$\Rightarrow$ f is analytic on (a,b).

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#10 f is analytic on $(-\infty,\infty) \Rightarrow f$ is conti. on $(-\infty,\infty)$.

$$\int_a^b |f(x)|\,dx = 0$$

$\Rightarrow f \equiv 0$ on $[a,b]$ by <u>§5.1 Ex 4 (b)</u>.

By <u>Thm 7.58</u>, f & 0 are both analytic on $\mathbb{R}$,

and $f=0$ on (a,b). $\therefore f \equiv 0$ on $\mathbb{R}$.

#11. It's easy to see that for each $n \in \mathbb{N}$,

$$\left(\sum_{k=1}^{n} |a_k|^\beta\right)^{1/\beta} \le \sum_{k=1}^{n} |a_k|$$

if $a_k \in \mathbb{R}$, $\beta > 1$. (Check by yourself).

$$\therefore \left(\sum_{k=1}^{n} |a_k|^\beta\right)^{1/\beta} \le \sum_{k=1}^{n} |a_k| \le \sum_{k=1}^{\infty} |a_k| \quad \forall n \in \mathbb{N}$$

$$\left(\sum_{k=1}^{n} |a_k|^\beta\right)^{1/\beta} \nearrow \left(\sum_{k=1}^{\infty} |a_k|^\beta\right)^{1/\beta} \quad \text{as } n \to \infty$$

$$\therefore \left(\sum_{k=1}^{\infty} |a_k|^\beta\right)^{1/\beta} \le \sum_{k=1}^{\infty} |a_k|.$$