§7.1

#1. Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\frac{|a|+|b|}{N} < \varepsilon$.

Then if $n \geq N$, $x \in [a,b]$,

$$|x| \leq |a| + |b|$$

$$\Rightarrow \left|\frac{x}{n}\right| \leq \frac{|a|+|b|}{n} \leq \frac{|a|+|b|}{N} < \varepsilon.$$

i.e. $\frac{x}{n} \longrightarrow 0$ as $n \longrightarrow \infty$ uniformly.

#4(a) g is conti. on $[a,b] \Rightarrow |g|$ is conti. on $[a,b]$

$\Rightarrow \exists x_0 \in [a,b]$ s.t. $0 < |g(x_0)| \leq |g(x)|$ $\forall x \in [a,b]$.

$g_n \longrightarrow g$ as $n \longrightarrow \infty$ uniformly

$\therefore$ For $\frac{|g(x_0)|}{2} > 0$, $\exists N_0 \in \mathbb{N}$ s.t.

$$|g_n(x) - g(x)| < \frac{|g(x_0)|}{2} \quad \text{if } x \in [a,b],\ n \geq N_0.$$

$$\therefore \quad |g_n(x)| \geq |g(x)| - |g_n(x) - g(x)|$$

$$> |g(x_0)| - \frac{|g(x_0)|}{2} = \frac{|g(x_0)|}{2} > 0$$

if $x \in [a,b]$, $n \geq N_0$.

$\therefore$ $\frac{1}{g_n(x)}$ is defined for $x \in [a,b]$ if $n \geq N_0$.

$\left(\frac{1}{|g_n(x)|} < \frac{2}{|g(x_0)|} := m \quad \text{if } x \in [a,b],\ n \geq N_0\right)$.

f is conti. on $[a,b] \Rightarrow \exists M > 0$ s.t.

$$|f(x)| \leq M \quad \forall x \in [a,b].$$

$f_n \to f$ & $g_n \to g$ as $n \to \infty$ uniformly

$\therefore$ Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ with $N \geq N_0$ s.t.

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2m}$$

$$|g_n(x) - g(x)| < \frac{\varepsilon}{m^2 M} \quad \text{if } x \in [a,b],\ n \geq N$$

$\therefore$ If $n \geq N \geq N_0$ and $x \in [a,b]$,

$$\left|\frac{f_n}{g_n}(x) - \frac{f}{g}(x)\right|$$

$$\leq \left|\frac{f_n(x)}{g_n(x)} - \frac{f(x)}{g_n(x)}\right| + \left|\frac{f(x)}{g_n(x)} - \frac{f(x)}{g(x)}\right|$$

$$= \frac{|f_n(x) - f(x)|}{|g_n(x)|} + \frac{|f(x)|}{|g_n(x)||g(x)|} |g_n(x) - g(x)|$$

$$< \frac{\varepsilon}{2m} \cdot m + M \cdot m \cdot \frac{m}{2} \cdot \frac{\varepsilon}{m^2 M} \quad \left(\begin{array}{l} \frac{1}{|g(x)|} \leq \frac{1}{|g(x_0)|} = \frac{m}{2} \\ \frac{1}{|g_n(x)|} \leq m \text{ since } n \geq N_1 \end{array} \right)$$

$$= \varepsilon$$

i.e. $\dfrac{f_n}{g_n} \longrightarrow \dfrac{f}{g}$ as $n \to \infty$ uniformly.

#4(b). Counterexample:

Let $f_n \equiv f \equiv 1$, $g_n(x) := x + \frac{1}{n}$, $g(x) := x$ on $(0,1)$.

Then $f_n \to f$, $g_n \to g$ as $n \to \infty$ uniformly.

And $|g(x)| = x > 0$ $\forall x \in (0,1)$.

But $\left|\frac{f_n}{g_n}\left(\frac{1}{n}\right) - \frac{f}{g}\left(\frac{1}{n}\right)\right| = \left|\frac{n}{2} - n\right| = \frac{n}{2} \geq \frac{1}{2}$ $\forall n \in \mathbb{N}$.

$\therefore \dfrac{f_n}{g_n}$ can't converge to $\dfrac{f}{g}$ uniformly as $n \to \infty$.

§7.1

#7. $\exists N \in \mathbb{N}$ s.t. $[a,b] \subset [-N, N]$.

By Bernoulli's ineq., $\because \frac{x}{n} \geq \frac{-a}{n} \geq \frac{-a}{N} \geq -1$ if $n \geq N$

$$\therefore \left(1+\frac{x}{n}\right)^{\frac{n}{n+1}} \leq 1 + \frac{x}{n}\cdot\frac{n}{n+1}$$

$$= 1 + \frac{x}{n+1}$$

i.e. $\left(1+\frac{x}{n}\right)^n \leq \left(1+\frac{x}{n+1}\right)^{n+1}$

$\therefore \left(1+\frac{x}{n}\right)^n \nearrow$ for $n \geq N$.

Moreover,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^{\lim_{n\to\infty} \ln\left(1+\frac{x}{n}\right)^n} = e^{\lim_{n\to\infty} n\log\left(1+\frac{x}{n}\right)}$$

$$= e^{\lim_{n\to\infty} \frac{\log\left(1+\frac{x}{n}\right)}{\frac{1}{n}}} = e^{\lim_{n\to\infty} \frac{\frac{1}{1+\frac{x}{n}}\cdot\frac{-x}{n^2}}{-\frac{1}{n^2}}} = e^x.$$

Lemma: Suppose that f_n is continuous on a bounded interval $[a,b]$ for each n, and $f_n \downarrow 0$ pointwise on $[a,b]$. Then $f_n \downarrow 0$ uniformly on $[a,b]$.

Assuming the Lemma, consider $f_n(x) = e^x - \left(1+\frac{x}{n}\right)^n$ on $[a,b]$, then $f_n \downarrow 0$ pointwise on $[a,b]$ for $n \geq N$. $\therefore f_n \downarrow 0$ as $n\to\infty$ uniformly on $[a,b]$.

i.e. $\left(1+\frac{x}{n}\right)^n \to e^x$ as $n\to\infty$ uniformly on $[a,b]$.

$\therefore$ By Thm 7.10, ($\left(1+\frac{x}{n}\right)^n e^{-x} \to e^x e^{-x} = 1$ uniformly)

$$\lim_{n\to\infty}\int_a^b \left(1+\frac{x}{n}\right)^n e^{-x}\,dx = \int_a^b dx = b-a.$$

pf of Lemma : Obviously, $f_n \geq 0$ on $[a,b]$

f_n is conti. on $[a,b]$

$\Rightarrow \exists\ x_n \in [a,b]$ s.t. $\max\limits_{x\in[a,b]} f_n(x) = f_n(x_n) := a_n$.

It's easy to see that $a_n \downarrow$ as $n \nearrow$.

$(\because\ a_{n+1} = f_{n+1}(x_{n+1}) \leq f_n(x_{n+1}) \leq f_n(x_n) = a_n)$

And $a_n \geq 0$. $\therefore$ $a_n \downarrow a$ as $n \to \infty$ for some $a \geq 0$.

$\{x_n\}_{n=1}^{\infty} \subset [a,b]$ and $[a,b]$ is sequentially compact.

$\therefore \exists$ a subseq. $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$, $x_0 \in [a,b]$,

s.t. $x_{n_j} \longrightarrow x_0$ as $j \to \infty$.

For each $k \in \mathbb{N}$ fixed, since $f_n \downarrow$

$$\therefore\ f_k(x_{n_j}) \geq f_{n_j}(x_{n_j}) \quad \text{for all } n_j \geq k$$

$$\begin{array}{ccc} f_k(x_{n_j}) & \geq & f_{n_j}(x_{n_j}) \\ \Big\downarrow {\scriptstyle \text{as } j\to\infty} & & \| \\ f_k(x_0) & & a_{n_j} \\ & & \Big\downarrow {\scriptstyle \text{as } j\to\infty} \\ & & a \end{array}$$

$\therefore$ $f_k(x_0) \geq a \quad \forall k \in \mathbb{N}$

Let $k \to \infty$, we get $0 \geq a$

$\therefore$ $a = 0$

i.e. $\max\limits_{x\in[a,b]} f_n(x) \downarrow 0$ as $n \longrightarrow \infty$.

$\Rightarrow f_n \downarrow 0$ as $n \longrightarrow \infty$ uniformly on $[a,b]$.

§7.1

#8. f_n is uniformly bounded on $[a,b]$. f is bounded on $[a,b]$.
g is continuous on $[a,b]$

$\therefore \exists M>0$ s.t.

$$|f_n(x)|\le M,\ |f(x)|\le M,\ |g(x)|\le M \quad \forall x\in[a,b]\ \forall n\in\mathbb{N}.$$

Given $\varepsilon>0$, g is continuous at a & b.

$\therefore \exists \delta>0$, s.t.

$$|g(x)-g(a)|<\frac{\varepsilon}{2M} \quad \text{if} \quad x\in[a,b]\cap[a,a+\delta)$$

$$|g(x)-g(b)|<\frac{\varepsilon}{2M} \quad \text{if} \quad x\in[a,b]\cap(b-\delta,b].$$

i.e. $$|g(x)|<\frac{\varepsilon}{2M} \quad \text{if} \quad x\in[a,b]\cap([a,a+\delta)\cup(b-\delta,b])$$

Choose $c,d\in[a,b]$ s.t. $a<c<a+\delta$, $b-\delta<d<b$, $c<d$.

On $[c,d]\subset(a,b)$, $f_n\longrightarrow f$ uniformly as $n\longrightarrow\infty$.

$\therefore \exists N\in\mathbb{N}$ s.t.

$$|f_n(x)-f(x)|<\frac{\varepsilon}{M} \quad \forall x\in[c,d],\ \text{if } n\ge N.$$

$\therefore$ If $n\ge N$, $x\in[a,b]$,

<u>Case $x\in[a,b]\setminus[c,d]$</u>:

$$|f_n(x)g(x)-f(x)g(x)|\le(|f_n(x)|+|f(x)|)|g(x)|$$
$$<(M+M)\cdot\frac{\varepsilon}{2M}=\varepsilon$$

<u>Case $x\in[a,b]$</u>:

$$|f_n(x)g(x)-f(x)g(x)|=|f_n(x)-f(x)||g(x)|$$
$$<\frac{\varepsilon}{M}\cdot M=\varepsilon$$

$\therefore f_ng\longrightarrow fg$ as $n\longrightarrow\infty$ uniformly on $[a,b]$.

§7.1 #10 f_n is bounded on E. $\therefore \exists M_n > 0$ s.t. $|f_n(x)| \le M_n \ \forall x \in E$.

1° f is bounded on E.

For $1 > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < 1 \quad \forall x \in E \text{ if } n \ge N$$

$\therefore |f(x)| < 1 + |f_N(x)| \le 1 + M_N < \infty$

$\therefore f$ is bounded by $M = 1 + M_N$

2° Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \forall x \in E \quad \text{if } n \ge N.$$

$\therefore$ For any $n \ge N$,

$$\left| \frac{f_1(x) + \cdots + f_n(x)}{n} - f(x) \right|$$

$$\le \frac{\left|\sum_{k=1}^{N-1} f_k(x)\right|}{n} + \frac{\sum_{k=N}^{n} |f_k(x) - f(x)|}{n} + \frac{N-1}{n}|f(x)|$$

$$< \frac{\sum_{k=1}^{N-1} M_k}{n} + \frac{n-N+1}{n}\frac{\varepsilon}{2} + \frac{N-1}{n} \cdot M$$

$$< \frac{\sum_{k=1}^{N-1} M_k + (N-1)M}{n} + \frac{\varepsilon}{2}. \qquad \forall x \in E.$$

$\exists \ \tilde{N} \in \mathbb{N}, \ \tilde{N} \ge N$ s.t.

$$\frac{\sum_{k=1}^{N-1} M_k + (N-1)M}{n} < \frac{\varepsilon}{2} \quad \text{if } n \ge \tilde{N}$$

Then if $n \ge \tilde{N}$, then

$$\left| \frac{f_1(x) + \cdots + f_n(x)}{n} - f(x) \right| < \varepsilon \quad \forall x \in E$$

i.e. $\dfrac{f_1(x) + \cdots + f_n(x)}{n} \longrightarrow f(x)$ uniformly on E as $n \to \infty$

§7.2

#6. $\left|\sum_{k=1}^{\infty}(1-\cos\frac{1}{k})\right| \le \sum_{k=1}^{\infty}\left|1-\cos\frac{1}{k}\right|$

$= \sum_{k=1}^{\infty}\left|\cos 0-\cos\frac{1}{k}\right| = \sum_{k=1}^{\infty}\left|-\sin(\frac{\theta_k}{k})(0-\frac{1}{k})\right|$

$\nearrow$ MVT where $\theta_k \in (0,1)$.

$\le \sum_{k=1}^{\infty}\frac{1}{k}\sin\frac{1}{k}$ ($\because \sin x \nearrow$ in $(0,1)$)

$\le \sum_{k=1}^{\infty}\frac{1}{k^2}$ ($\because \sin x \le x$ in $(0,1)$)

$= 1+\sum_{k=2}^{\infty}\frac{1}{k^2} \le 1+\int_1^{\infty}\frac{1}{x^2}dx = 2$.

#7. g_1 is bounded on E.

$\therefore \exists M>0$ s.t. $|g_1(x)|<M \quad \forall x\in E$

$\therefore 0\le |g_k(x)| = g_k(x) \le g_1(x) < M \quad \forall x\in E,\ k\in\mathbb{N}$.

$f=\sum_{k=1}^{\infty} f_k$ converges uniformly on E.

$\therefore$ Given $\varepsilon>0$, $\exists N\in\mathbb{N}$ s.t.

$$\left|\sum_{k=n+1}^{m} f_k(x)\right| < \frac{\varepsilon}{M} \quad \forall x\in E,\ m>n\ge N.$$

$\therefore \left|\sum_{k=n+1}^{m} f_k(x)g_k(x)\right|$

$= \left|\left(\sum_{j=n+1}^{m} f_j(x)\right)g_m(x) + \sum_{k=n+1}^{m-1}\left(\sum_{j=n+1}^{k} f_j(x)\right)(g_k(x)-g_{k+1}(x))\right|$

$\le \left|\sum_{j=n+1}^{m} f_j(x)\right| g_m(x) + \sum_{k=n+1}^{m-1}\left|\sum_{j=n+1}^{k} f_j(x)\right|(g_k(x)-g_{k+1}(x))$

$\le \frac{\varepsilon}{M}\, g_m(x) + \sum_{k=n+1}^{m-1}\frac{\varepsilon}{M}(g_k(x)-g_{k+1}(x))$

$= \frac{\varepsilon}{M}\, g_{n+1}(x) < \frac{\varepsilon}{M}\cdot M = \varepsilon \quad \forall x\in E,\ m>n\ge N$.

$\therefore \sum_{k=1}^{\infty} f_k(x)g_k(x)$ converges uniformly on E.

§7.3

#2. (a) For $x \in (-1, 1)$,

$$\sum_{k=1}^{\infty} 3x^{3k-1} = \frac{3x^2}{1-x^3}.$$

(b) For $x \in (-1, 1)$

$$\sum_{k=2}^{\infty} k x^{k-2} = \sum_{k=2}^{\infty} (k-1) x^{k-2} + \sum_{k=2}^{\infty} x^{k-2}$$

$$= \sum_{k=2}^{\infty} (x^{k-1})' + \sum_{k=2}^{\infty} x^{k-2} = \left(\sum_{k=2}^{\infty} x^{k-1}\right)' + \sum_{k=2}^{\infty} x^{k-2}$$

$$= \left(\frac{x}{1-x}\right)' + \frac{1}{1-x} = \frac{2-x}{(1-x)^2}$$

(c) For $x \in (0, 2)$,

$$\sum_{k=1}^{\infty} \frac{2k}{k+1} (1-x)^k = \sum_{k=1}^{\infty} 2(1-x)^k - \sum_{k=1}^{\infty} \frac{2}{k+1} (1-x)^k$$

$$= \begin{cases} \dfrac{2(1-x)}{1-(1-x)} + \dfrac{2}{1-x} \displaystyle\int_1^x \frac{1-t}{1-(1-t)}\, dt & , x \neq 1 \\ \dfrac{2(1-x)}{1-(1-x)} = 0 & , x = 1 \end{cases}$$

$$= \begin{cases} \dfrac{2 \ln x}{1-x} + \dfrac{2}{x} & \text{if } x \neq 1 \\ \dfrac{2(1-x)}{x} = 0 & \text{if } x = 1 \end{cases}$$

(d) For $y \in (-1, 1)$

$$\sum_{k=0}^{\infty} \frac{y^k}{k+1} = \begin{cases} \dfrac{1}{y} \displaystyle\int_0^y \frac{1}{1-t}\, dt, & \text{if } y \neq 0 \\ 1 & \text{if } y = 0. \end{cases}$$

$$= \begin{cases} -\dfrac{\ln(1-y)}{y} & \text{if } y \neq 0 \\ 1 & \text{if } y = 0 \end{cases}$$

$\therefore$ For $x \in (-1, 1)$, $x^3 \in (-1, 1)$

$$\sum_{k=0}^{\infty} \frac{x^{3k}}{k+1} = \begin{cases} -\dfrac{\ln(1-x^3)}{x^3} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

§7.3

#6. Suppose that $|a_k| \leq M$ $\forall k \in \mathbb{N} \cup \{0\}$ for some $M>0$

(a) For each x with $|x|<1$,

$$\limsup_{k\to\infty} \sqrt[k]{|a_k x^k|} \leq \limsup_{k\to\infty} \left(M^{\frac{1}{k}}|x|\right) = |x| < 1$$

$\therefore \sum_{k=0}^{\infty} a_k x^k$ converges absolutely for $|x|<1$

$\therefore f(x) = \sum_{k=0}^{\infty} a_k x^k$ has a radius R of convergence with $R \geq 1$.

(b). By Thm 7.27 (Abel's Thm),

$f(x) = \sum_{k=0}^{\infty} a_k x^k$ is conti. and converges uniformly on $[a-1, b]$

$\therefore f$ is uniformly continuous on $[a-1, b]$.

Given $\varepsilon>0$, $\exists\, \delta>0$ s.t.

$$|f(x)-f(y)|<\varepsilon \quad \text{if} \quad |x-y|<\delta \;\&\; x, y \in [a-1, b].$$

Choose $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \delta$, then $\forall x \in [a, b]$, $n \geq N$.

$$\left|x - \left(x - \tfrac{1}{n}\right)\right| = \tfrac{1}{n} \leq \tfrac{1}{N} < \delta$$

and $x,\ x - \frac{1}{n} \in [a-1, b]$.

$$\Rightarrow \left|f(x) - f\left(x-\tfrac{1}{n}\right)\right| < \varepsilon.$$

$$\qquad\qquad \| $$

$$|f(x) - f_n(x)|$$

i.e. $f_n \longrightarrow f$ uniformly on $[a, b]$.

§7.3

#7 (a) $\sum_{k=0}^{\infty} a_k$ converges $\Rightarrow \sum_{k=0}^{\infty} a_k r^k$ converges for $r \in [0,1]$.

Abel's Thm $\Rightarrow f(r) := \sum_{k=0}^{\infty} a_k r^k$ is conti. on $[0,1]$

$$\Rightarrow \lim_{r \to 1-} f(r) = f(1) = \sum_{k=0}^{\infty} a_k = L$$

$$\overset{\Vert}{\lim_{r\to 1-} \sum_{k=0}^{\infty} a_k r^k}$$

i.e. $\sum_{k=0}^{\infty} a_k$ is Abel summable to L.

(b). $$\sum_{k=0}^{\infty} (-1)^k r^k = \frac{1}{1-(-r)} = \frac{1}{1+r} \quad \text{for } r \in (-1,1)$$

$$\lim_{r\to 1-} \sum_{k=0}^{\infty} (-1)^k r^k = \lim_{r \to 1-} \frac{1}{1+r} = \frac{1}{2}.$$

§7.3

#8. For each $x \in (-3, 3)$,

$$\limsup_{k\to\infty} \left| \frac{x}{(-1)^k+4} \right| = \frac{|x|}{3} < 1$$

$\therefore \sum_{k=0}^{\infty} \left(\frac{x}{(-1)^k+4} \right)^k$ converges for $x \in (-3, 3)$

i.e. f has a radius of convergence greater than 3.

By Thm 7.30, f is differentiable on $(-3, 3)$, and

$$f'(x) = \sum_{k=1}^{\infty} \frac{k x^{k-1}}{((-1)^k+4)^k}.$$

For $0 \le x < 3$,

$$\begin{aligned} |f'(x)| &= \sum_{k=1}^{\infty} \frac{k x^{k-1}}{((-1)^k+4)^k} \\ &\le \sum_{k=1}^{\infty} \frac{k x^{k-1}}{3^k} \qquad (\because (-1)^k+4 \ge 3 \quad \forall k \in \mathbb{N}) \\ &= \sum_{k=1}^{\infty} \frac{k}{3} \left(\frac{x}{3} \right)^{k-1} \\ &= \frac{3}{(3-x)^2}. \end{aligned}$$

$$\left(\begin{aligned} &\text{Note that } \sum_{k=1}^{\infty} \frac{k}{3} y^{k-1} = \sum_{k=1}^{\infty} \frac{1}{3} (y^k)' = \frac{1}{3} \left(\sum_{k=1}^{\infty} y^k \right)' \\ &= \frac{1}{3} \left(\frac{y}{1-y} \right)' = \frac{1}{3(1-y)^2} \quad \text{for } y \in (-1, 1). \\ &\frac{x}{3} \in (0, 1) \quad \therefore \sum_{k=1}^{\infty} \frac{k}{3} \left(\frac{x}{3} \right)^{k-1} = \frac{1}{3(1-\frac{x}{3})^2} = \frac{3}{(3-x)^2} \end{aligned} \right).$$

§7.3

#9 $a_k \searrow 0$ as $k \to \infty$

$\therefore \sum_{k=0}^{\infty} (-1)^k a_k$ converges.

$\therefore \sum_{k=0}^{\infty} (-1)^k a_k x^k$ converges on $[0,1]$.

Abel's Thm $\Rightarrow$ $f(x) := \sum_{k=0}^{\infty} (-1)^k a_k x^k$ is conti. on $[0,1]$.

$\therefore$ f is uniformly conti. on $[0,1]$.

$\Rightarrow$ Given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon \quad \text{if} \quad |x-y| < \delta \ \& \ x, y \in [0,1]$$

$$\shortparallel$$

$$\left| \sum_{k=0}^{\infty} (-1)^k a_k x^k - \sum_{k=0}^{\infty} (-1)^k a_k y^k \right|$$

$$\shortparallel$$

$$\left| \sum_{k=0}^{\infty} (-1)^k a_k (x^k - y^k) \right|.$$