§5.4

#7(a) Suppose $L>0$, then for $\frac{L}{2}>0$, $\exists M>0$ s.t.

$$|f(x)-L|<\frac{L}{2} \quad \text{if } x>M$$

i.e. $\frac{L}{2}<f(x)<\frac{3}{2}L$ if $x>M$

$\therefore$ For $M'>M$,

$$\begin{aligned}\int_0^{M'} f(x)\,dx &= \int_0^M f(x)\,dx + \int_M^{M'} f(x)\,dx \\ &\geq \int_0^M f(x)\,dx + \int_M^{M'} \frac{L}{2}\,dx \\ &= \int_0^M f(x)\,dx + \frac{L}{2}(M'-M) \longrightarrow \infty \quad \text{as } M'\to\infty\end{aligned}$$

This contradicts to that f is improporly integrable on $[0,\infty)$.

Hence $L\leq 0$.

Let $g=-f$. g is integrable on $[0,M']$ for any $M'>0$ since f is. And

$$\lim_{M'\to\infty}\int_0^{M'} g(x)\,dx = -\lim_{M'\to\infty}\int_0^{M'} f(x)\,dx \quad \text{exists.}$$

$\therefore$ g is improperly integrable on $[0,\infty)$.

$$\lim_{x\to\infty} g(x) = -\lim_{x\to\infty} f(x) = -L$$

By the previous result, $-L\leq 0 \Rightarrow L\geq 0$

$\therefore L=0$.

(b). Easy to check that f is integrable on $[0,b]$ for any $b>0$.

And $$\int_0^b f(x)\,dx = \begin{cases} \sum_{k=1}^{n} 2^{-k} & \text{if } n+2^{-n}\leq b<n+1 \\ \sum_{k=1}^{n-1} 2^{-k}+(b-n) & \text{if } n\leq b<n+2^{-n}\end{cases}$$

$$\leq \sum_{k=1}^{n} 2^{-k} < \sum_{k=1}^{\infty} 2^{-k} = 1.$$

$\int_0^b f(x)\,dx \nearrow$ as $b\nearrow\infty$, and $\int_0^b f(x)\,dx$ is bdd above by 1

$\therefore$ $\lim_{b\to\infty}\int_0^b f(x)\,dx$ exists. i.e. f is improperly integrable on $[0,\infty)$.

$n\to\infty$, $n+2^{-n}\to\infty$ but

$$\lim_{n\to\infty} f(n) = 1 \neq 0 = \lim_{n\to\infty} f(n+2^{-n}).$$

$\therefore$ $\lim_{x\to\infty} f(x)$ does not exist.

§5.4

\#8 f is absolutely integrable on $[1,\infty)$

$\therefore$ $|f|$ is integrable on $[1,b)$ for any $b>1$

and $\lim_{b\to\infty}\int_0^b |f(x)|\,dx$ exists.

For any $b>1$,

$$0 \le \int_1^b |f(x^n)|\,dx \overset{\text{Thm 5.34}}{=} \int_1^{b^n} \frac{|f(y)|}{n y^{\frac{n-1}{n}}}\,dy$$

$$\le \frac{1}{n}\int_1^{b^n} |f(y)|\,dy \le \frac{1}{n}\int_1^\infty |f(y)|\,dy$$

$$\left(\begin{array}{l} \because \int_1^{b^n} |f(y)|\,dy \nearrow \text{ as } b^n \nearrow \infty \\ \text{and } \int_1^\infty |f(y)|\,dy \text{ exists} \end{array}\right)$$

$\int_1^b |f(x^n)|\,dx \nearrow$ as $b \nearrow \infty$ and is bdd above by $\frac{1}{n}\int_1^\infty |f(y)|\,dy$

$\therefore$ $\lim_{b\to\infty}\int_1^b |f(x^n)|\,dx$ exists and

$$0 \le \int_1^\infty |f(x^n)|\,dx \le \frac{1}{n}\int_1^\infty |f(y)|\,dy$$

$$\lim_{n\to\infty}\frac{1}{n}\int_1^\infty |f(y)|\,dy = 0 \Rightarrow \lim_{n\to\infty}\int_1^\infty |f(x^n)|\,dx = 0.$$

$\Rightarrow$ $f(x^n)$ is improperly integrable on $[1,\infty)$ by Thm 5.48,

and

$$0 \le \left|\int_1^\infty f(x^n)\,dx\right| \le \int_1^\infty |f(x^n)|\,dx$$

$$\lim_{n\to\infty}\int_1^\infty |f(x^n)|\,dx = 0 \Rightarrow \lim_{n\to\infty}\int_1^\infty f(x^n)\,dx = 0.$$

10 (a) $\sin x \geq \frac{2}{\pi} x \quad \forall x \in [0, \frac{\pi}{2}]$

$$\left(\because \left(\frac{\sin x}{x}\right)' = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x}{x^2}(x - \tan x) < 0 \quad \forall x \in (0, \tfrac{\pi}{2}) \right.$$

$$\left. \therefore \frac{\sin x}{x} \searrow \text{ as } x \nearrow \Rightarrow \frac{\sin x}{x} \geq \frac{\sin \frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2}{\pi} \quad \forall x \in [0, \tfrac{\pi}{2}] \right)$$

$$\therefore \int_0^{\frac{\pi}{2}} e^{-a\sin x}\, dx \leq \int_0^{\frac{\pi}{2}} e^{-a\frac{2}{\pi}x}\, dx = -\frac{\pi}{2a} e^{-\frac{2a}{\pi}x}\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2a}(1 - e^{-a}) < \frac{\pi}{2a} < \frac{2}{a} \qquad (\because \pi < 4)$$

(b) $$\int_0^{\frac{\pi}{2}} e^{-a\sin x}\, dx = \int_0^{\frac{\pi}{2}} e^{-a\sin y}\, dy \leq \frac{2}{a}$$

$y = \frac{\pi}{2} - x$

$dy = -dx$

S15

E1 (a) $$\frac{4k}{4k^2-1} > \frac{4k}{4k^2} = \frac{1}{k} \qquad \forall k \in \mathbb{N}$$

(b) $$\sum_{k=1}^{2^n-1}\frac{1}{k} = \sum_{i=0}^{n-1}\left(\sum_{k=2^i}^{2^{i+1}-1}\frac{1}{k}\right) > \sum_{i=0}^{n-1}\left(\sum_{k=2^i}^{2^{i+1}-1}\frac{1}{2^{i+1}-1}\right)$$
$$= \sum_{i=0}^{n-1}\left(\frac{(2^{i+1}-1)-(2^i-1)}{2^{i+1}-1}\right)$$
$$= \sum_{i=0}^{n-1}\frac{2^i}{2^{i+1}-1} > \sum_{i=0}^{n-1}\frac{2^i}{2^{i+1}} = \sum_{i=0}^{n-1}\frac{1}{2} = \frac{n}{2} \qquad \forall n \in \mathbb{N}$$

(c) For each n, let $P_n = \left\{\frac{1}{\sqrt{(2k+\frac{1}{2})\pi}} \,\middle|\, 0 \le k \le n\right\} \cup \left\{\frac{1}{\sqrt{(2k+\frac{3}{2})\pi}} \,\middle|\, 0 \le k \le n\right\} \cup \{0,1\}$ be a partition of $[0,1]$. Then

$$V(\phi, P_n) = \sum_{k=0}^{n}\left|\phi\left(\frac{1}{\sqrt{(2k+\frac{1}{2})\pi}}\right) - \phi\left(\frac{1}{\sqrt{(2k+\frac{3}{2})\pi}}\right)\right|$$
$$+ \sum_{k=0}^{n-1}\left|\phi\left(\frac{1}{\sqrt{(2k+\frac{3}{2})\pi}}\right) - \phi\left(\frac{1}{\sqrt{(2k+\frac{5}{2})\pi}}\right)\right| +$$
$$+ \left|\phi(1) - \phi\left(\frac{1}{\sqrt{\frac{\pi}{2}}}\right)\right| + \left|\phi\left(\frac{1}{\sqrt{(2n+\frac{3}{2})\pi}}\right) - \phi(0)\right|$$
$$= \sum_{k=0}^{n}\left(\frac{1}{(2k+\frac{1}{2})\pi} + \frac{1}{(2k+\frac{3}{2})\pi}\right) + \sum_{k=0}^{n-1}\left(\frac{1}{(2k+\frac{3}{2})\pi} + \frac{1}{(2k+\frac{5}{2})\pi}\right)$$
$$+ \left|\sin 1 - \frac{1}{\frac{\pi}{2}}\right| + \left|\frac{1}{(2n+\frac{3}{2})\pi}\right|$$
$$> \sum_{k=1}^{n}\left(\frac{2}{(2k+\frac{1}{2})\pi} + \frac{2}{(2k+\frac{3}{2})\pi}\right)$$
$$= \frac{8}{\pi}\sum_{k=1}^{n}\frac{4(2k+1)}{4(2k+1)^2-1} > \frac{8}{\pi}\sum_{k=1}^{n}\frac{1}{2k+1} \quad \text{by (a)}$$
$$> \frac{8}{\pi}\sum_{k=1}^{n}\frac{1}{3k} = \frac{8}{3\pi}\sum_{k=1}^{n}\frac{1}{k}$$

$\therefore$ $V(\phi, P_{2^n-1}) > \frac{8}{3\pi}\sum_{k=1}^{2^n-1}\frac{1}{k} > \frac{8}{3\pi}\frac{n}{2} \longrightarrow \infty$ as $n \longrightarrow \infty$

$\therefore$ $\{V(\phi, P) \mid P$ is a partition of $[0,1]\}$

$\Rightarrow$ $Var(\phi) = \infty$ i.e. ϕ is not of bounded variation on $[0,1]$

#2 (a) $\dfrac{8k^2+2}{(4k^2-1)^2} < \dfrac{1}{k^2}$ for $k \geq 2$

$\Leftrightarrow (8k^2+2)k^2 < (4k^2-1)^2 = 16k^4 - 8k^2 + 1$ for $k \geq 2$

$\underset{8k^4+16k^2}{\overset{\parallel}{(8k^2+2)k^2}}$

$\Leftrightarrow 8k^4 - 24k^2 + 1 > 0$

But $8k^4 - 24k^2 + 1 = 8(k^4 - 3k^2 + \frac{9}{4}) + 1 - 18$

$$= 8(k^2 - \tfrac{3}{2})^2 - 17$$

$$\geq 8(2^2 - \tfrac{3}{2})^2 - 17$$

$$= 33 > 0 \qquad \forall k \geq 2$$

$\therefore \dfrac{8k^2+2}{(4k^2-1)^2} < \dfrac{1}{k^2} \quad \forall k \geq 2.$

(b) $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2} \leq \sum_{k=1}^{n-1} \frac{1}{k^2+k} + \frac{1}{n^2} \leq \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) + 1 = \left(1 - \frac{1}{n}\right) + 1 = 2 - \frac{1}{n}$

(c) Note that ϕ is diff. on $\mathbb{R}$ and

$$\phi'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

For any bdd. interval $[a, b]$

$$|\phi'(x)| \leq 2 \max\{|a|, |b|\} + 1 = M \qquad \forall x \in [a, b].$$

If $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a,b]$, then

$$\sum_{j=1}^{n} |\phi(x_j) - \phi(x_{j-1})|$$

$$= \sum_{j=1}^{n} |\phi'(c_j)||x_j - x_{j-1}| \quad \text{for some } c_j \in (x_{j-1}, x_j) \text{ by MVT}$$

$$\leq \sum_{j=1}^{n} M(x_j - x_{j-1}) = M(b-a) < \infty$$

$\therefore \sup\{V(\phi, P) \mid P \text{ is a partition of } [a,b]\} \leq M(b-a) < \infty$

i.e. ϕ is of bounded variation on $[a, b]$.

§5.5. #4

($\Leftarrow$) Obvious.

($\Rightarrow$) ϕ is of bounded variation.

$\Rightarrow \phi$ is bdd. and $\phi = f - g$ for some increasing fcns f & g on $[a,b]$.

By Lemma 4.28,

$$f(a+),\ g(a+),\ f(b-),\ g(b-) \text{ exist}$$

and
$$f(a) \le f(a+) \le f(b-) \le f(b)$$
$$g(a) \le g(a+) \le g(b-) \le g(b).$$

$\therefore$ $\phi(a+) = f(a+) - g(a+)$, $\phi(b-) = f(b-) - g(b-)$ exist and are finite.

i.e. ϕ can be continuously extended to $[a,b]$.

$\Rightarrow \phi$ is uniformly conti. on (a,b).

§5.6

#2. Fix $x, y \in I$, $\alpha \in [0,1]$.

f_n is convex $\Rightarrow$ $f_n(\alpha x+(1-\alpha)y) \le \alpha f_n(x) + (1-\alpha) f_n(y)$

$$\therefore \underbrace{\lim_{n\to\infty} f_n(\alpha x+(1-\alpha)y)}_{= f(\alpha x+(1-\alpha)y)} \le \underbrace{\lim_{n\to\infty} (\alpha f_n(x) + (1-\alpha) f_n(y))}_{= \alpha f(x) + (1-\alpha) f(y)}$$

(Note that, $\alpha x+(1-\alpha)y, x, y \in I$)

$\therefore f$ is convex on I.

#6. Let $y = \dfrac{x-a}{b-a}$ $\quad \dfrac{dy}{dx} = \dfrac{1}{b-a}$

f is integrable on $[a,b]$
$\Rightarrow f$ is bdd. on $[a,b]$.
May assume that $f:[a,b] \to [c,d]$ for some bounded interval $[c,d]$

By Thm 5.34

$$\left(\int_a^b |f(x)|\,dx\right)^2 = \left(\int_0^1 |f(a+(b-a)y)|(b-a)\,dy\right)^2$$

$$= (b-a)^2 \left(\int_0^1 |f(a+(b-a)y)|\,dy\right)^2$$

Jenssen's ineq. with $\phi:[c,d]\to\mathbb{R}$, $\phi(t)=t^2$

$$\le (b-a)^2 \int_0^1 |f(a+(b-a)y)|^2\,dy$$

$$\overset{\text{Thm 5.34}}{=} (b-a)^2 \int_a^b |f(x)|^2 \frac{1}{b-a}\,dx$$

$$= (b-a)\int_a^b |f(x)|^2\,dx$$

i.e. $\displaystyle\int_a^b |f(x)|\,dx = (b-a)^{1/2}\left(\int_a^b |f(x)|^2\,dx\right)^{1/2}$

#7. (a). e^x is convex in $\mathbb{R}$

$\therefore e^{\int_0^1 f(x)\,dx} \le \displaystyle\int_0^1 e^{f(x)}\,dx$ by Jenssen's ineq.

$|x|^{\frac{1}{r}}$ is convex in $\mathbb{R}$

$\therefore \left(\displaystyle\int_0^1 |f(x)|\,dx\right)^{\frac{1}{r}} \le \displaystyle\int_0^1 |f(x)|^{\frac{1}{r}}\,dx$ by Jenssen's ineq.

$\therefore \displaystyle\int_0^1 |f(x)|\,dx \le \left(\int_0^1 |f(x)|^{\frac{1}{r}}\,dx\right)^r$.

#7(b). Apply Jenssen's ineq. with $|f|^p$ and convex fcn $|\cdot|^{\frac{q}{p}}$, we get

$$\left(\int_0^1 |f(x)|^p dx\right)^{\frac{q}{p}} \le \int_0^1 \left(|f(x)|^p\right)^{\frac{q}{p}} dx$$
$$= \int_0^1 |f(x)|^q dx$$

$$\Rightarrow \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}} \le \left(\int_0^1 |f(x)|^q dx\right)^{\frac{1}{q}}$$

(Note that $|\cdot|^{\frac{q}{p}}$ is convex since $q > p > 0$, and $|f|^p$, $|f|^q$ are integrable on $[0,1]$).

#7(c) ① If $f : [a,b] \longrightarrow [c,d]$ is integrable, where $[a,b]$, $[c,d]$ are finite intervals, $\phi : [c,d] \longrightarrow \mathbb{R}$ is convex and $\phi \circ f$ is integrable on $[a,b]$, then

$$\phi\left(\frac{\int_a^b f(x)\,dx}{b-a}\right) \le \frac{1}{b-a}\int_a^b (\phi\circ f)(x)\,dx.$$

pf: $\phi\left(\frac{\int_a^b f(x)\,dx}{b-a}\right) = \phi\left(\int_0^1 f(a+(b-a)y)\,dy\right)$ $\quad y = \frac{x-a}{b-a}$, $dy = \frac{1}{b-a}dx$

$$\le \int_0^1 (\phi\circ f)(a+(b-a)y)\,dy$$
$$= \frac{1}{b-a}\int_a^b (\phi\circ f)(x)\,dx.$$

② If $f : (a,b) \longrightarrow \mathbb{R}$ is improperly integrable, a, b are finite, and $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is convex, $\phi\circ f$ is improperly integrable on (a,b), then

$$\phi\left(\frac{\int_a^b f(x)\,dx}{b-a}\right) \le \frac{1}{b-a}\int_a^b (\phi\circ f)(x)\,dx.$$

pf: $f : [a', b']$ is integrable for any $[a', b'] \subset (a,b)$.

$\therefore f$ is bdd. on $[a', b']$, say f is bdd. by M

$\therefore f : [a', b'] \longrightarrow [-M, M]$ is integrable

& $\phi : [-M, M] \longrightarrow \mathbb{R}$ is convex

$\phi\circ f$ is integrable on $[a', b']$

$\therefore$ By ①,

$$\phi\left(\frac{\int_{a'}^{b'} f(x)\,dx}{b'-a'}\right) \le \frac{1}{b'-a'}\int_{a'}^{b'}(\phi\circ f)(x)\,dx$$

$$\frac{\int_{a'}^{b'} f(x)\,dx}{b'-a'} \longrightarrow \frac{\int_a^b f(x)\,dx}{b-a} \quad \text{as} \quad \begin{array}{l} b'\to b- \\ a'\to a+ \end{array}$$

& ϕ is convex $\Rightarrow$ ϕ is conti.

$$\therefore \phi\left(\frac{\int_{a'}^{b'} f(x)\,dx}{b'-a'}\right) \longrightarrow \phi\left(\frac{\int_a^b f(x)\,dx}{b-a}\right)$$

as $b'\to b-$, $a'\to a+$

$$\frac{1}{b'-a'}\int_{a'}^{b'}(\phi\circ f)(x)\,dx \longrightarrow \frac{1}{b-a}\int_a^b(\phi\circ f)(x)\,dx$$

as $b'\to b-$, $a'\to a+$

$$\therefore \phi\left(\frac{\int_a^b f(x)\,dx}{b-a}\right) \le \frac{1}{b-a}\int_a^b(\phi\circ f)(x)\,dx.$$

Back to #7(c),

if $f:[0,1]\to\mathbb{R}$ is improperly integrable, and $e^{f(x)}$, $|f(x)|^p$ are improperly integrable on $(0,1)$ $\forall\, 0<p<\infty$,

then $$e^{\int_0^1 f(x)\,dx} \le \int_0^1 e^{f(x)}\,dx$$

$$\left(\int_0^1 |f(x)|^r\,dx\right)^{1/r} \le \int_0^1 |f(x)|\,dx \quad \text{for } 0<r\le 1$$

$$\left(\int_0^1 |f(x)|^p\,dx\right)^{1/p} \le \left(\int_0^1 |f(x)|^q\,dx\right)^{1/q} \quad \text{for } 0<p<q.$$

By ②, it's obvious.