

35.1

#2 (a) Obvious.

(b). For each $n \in \mathbb{N}$,

$$L(f, P_n) \leq (L) \int_a^b f(x) dx \leq (U) \int_a^b f(x) dx \leq U(f, P_n)$$

Rmk 5.14

$$\lim_{n \rightarrow \infty} L(f, P_n) = \lim_{n \rightarrow \infty} U(f, P_n) = I_0$$

$$\Rightarrow (L) \int_a^b f(x) dx = (U) \int_a^b f(x) dx = I_0$$

Thm 5.15 \Rightarrow f is integrable on $[0, 1]$ and

$$\int_0^1 f(x) dx = I_0$$

(c) (a) $f(x) = x$ $f \nearrow$

Fix $n \in \mathbb{N}$,
 $\sup_{x \in [\frac{j-1}{n}, \frac{j}{n}]} f(x) = f\left(\frac{j}{n}\right) = \frac{j}{n}$ for $j=1, 2, \dots, n$.

$$\inf_{x \in [\frac{j-1}{n}, \frac{j}{n}]} f(x) = f\left(\frac{j-1}{n}\right) = \frac{j-1}{n}$$

$$\therefore U(f, P_n) = \sum_{j=1}^n \frac{j}{n} \left(\frac{j}{n} - \frac{j-1}{n} \right) = \sum_{j=1}^n \frac{j}{n^2} = \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{n+1}{2n}$$

$$L(f, P_n) = \sum_{j=1}^n \frac{j-1}{n} \left(\frac{j}{n} - \frac{j-1}{n} \right) = \sum_{j=1}^n \frac{j-1}{n^2} = \frac{1}{n^2} \left[\frac{n(n+1)}{2} - n \right] = \frac{n-1}{2n}$$

$$\lim_{n \rightarrow \infty} U(f, P_n) = \frac{1}{2} = \lim_{n \rightarrow \infty} L(f, P_n)$$

(b) \Rightarrow f is integrable on $[0, 1]$ and $\int_0^1 f(x) dx = \frac{1}{2}$

(b) $f(x) = x^2$ $f \nearrow$

Fix $n \in \mathbb{N}$,

$$\sup_{x \in [\frac{j-1}{n}, \frac{j}{n}]} f(x) = f\left(\frac{j}{n}\right) = \frac{j^2}{n^2}$$

$$\inf_{x \in [\frac{j-1}{n}, \frac{j}{n}]} f(x) = f\left(\frac{j-1}{n}\right) = \frac{(j-1)^2}{n^2}$$

$$\therefore \mathbb{U}(f, P_n) = \sum_{j=1}^n \frac{j^2}{n^2} \left(\frac{j}{n} - \frac{j-1}{n} \right) = \sum_{j=1}^n \frac{j^2}{n^3} = \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

$$\begin{aligned} L(f, P_n) &= \sum_{j=1}^n \frac{(j-1)^2}{n^2} \left(\frac{j}{n} - \frac{j-1}{n} \right) = \sum_{j=1}^n \frac{(j-1)^2}{n^3} = \frac{1}{n^3} \left[\frac{n(n+1)(2n+1)}{6} - 2 \frac{n(n+1)}{2} + n \right] \\ &= \frac{(2n-1)(n-1)}{6n^2} \end{aligned}$$

$$\lim_{n \rightarrow \infty} \mathbb{U}(f, P_n) = \frac{1}{3} = \lim_{n \rightarrow \infty} L(f, P_n)$$

$\stackrel{(b)}{\Rightarrow} f$ is integrable on $[0, 1]$ and $\int_0^1 f(x) dx = \frac{1}{3}$.

$$(Y) f(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ 2 & \frac{1}{2} \leq x \leq 1 \end{cases} \quad f \nearrow$$

Fix $n \in \mathbb{N}$,

$$\sup_{x \in [\frac{j-1}{n}, \frac{j}{n}]} f(x) = f\left(\frac{j}{n}\right) = \begin{cases} 1 & \text{if } j < \frac{n}{2} \\ 2 & \text{if } j \geq \frac{n}{2} \end{cases}$$

$$\inf_{x \in [\frac{j-1}{n}, \frac{j}{n}]} f(x) = f\left(\frac{j-1}{n}\right) = \begin{cases} 1 & \text{if } j-1 < \frac{n}{2} \\ 2 & \text{if } j-1 \geq \frac{n}{2} \end{cases}$$

$$\therefore \mathbb{U}(f, P_n) = \sum_{j=1}^n f\left(\frac{j}{n}\right) \left(\frac{j}{n} - \frac{j-1}{n} \right) = \sum_{j=1}^{\lceil \frac{n}{2} \rceil} \frac{1}{n} + \sum_{j=\lceil \frac{n}{2} \rceil + 1}^n \frac{2}{n} = 2 - \frac{\lceil \frac{n}{2} \rceil}{n}$$

$$L(f, P_n) = \sum_{j=1}^n f\left(\frac{j-1}{n}\right) \left(\frac{j}{n} - \frac{j-1}{n} \right) = \sum_{j=1}^{\lceil \frac{n}{2} \rceil} \frac{1}{n} + \sum_{j=\lceil \frac{n}{2} \rceil + 1}^n \frac{2}{n} = 2 - \frac{\lceil \frac{n}{2} \rceil}{n}$$

$$\left(\frac{n-3}{2n} = \frac{\frac{n-1}{2}-1}{n} < \frac{\lceil \frac{n-1}{2} \rceil}{n} \leq \frac{\lceil \frac{n}{2} \rceil}{n} \leq \frac{n}{n} = \frac{1}{2} \right)$$

$$\left(\lim_{n \rightarrow \infty} \frac{n-3}{2n} = \frac{1}{2} \Rightarrow \lim_{n \rightarrow \infty} \frac{\lceil \frac{n-1}{2} \rceil}{n} = \frac{\lceil \frac{n}{2} \rceil}{n} = \frac{1}{2} \right)$$

$$\therefore \lim_{n \rightarrow \infty} \mathbb{U}(f, P_n) = \lim_{n \rightarrow \infty} L(f, P_n) = 2 - \frac{1}{2} = \frac{3}{2}$$

$\stackrel{(b)}{\Rightarrow} f$ is integrable on $[0, 1]$ and $\int_0^1 f(x) dx = \frac{3}{2}$.

35.1 #3 For each $n \in \mathbb{N}$, let $P_n = \{0, \frac{1}{n}\} \cup \left\{ \frac{1}{n} + \frac{j}{2^n}(1 - \frac{1}{n}) \mid j=1, 2, \dots, 2^n \right\}$, a partition of $[0, 1]$

$$\text{Then } U(f, P_n) = \left(\sup_{x \in [0, \frac{1}{n}]} f(x) \right) \left(\frac{1}{n} - 0 \right) + \sum_{j=1}^{2^n} M_j \frac{1}{2^n}$$

$$\text{where } M_j = \sup_{x \in [\frac{1}{n} + \frac{j-1}{2^n}(1 - \frac{1}{n}), \frac{1}{n} + \frac{j}{2^n}(1 - \frac{1}{n})]} f(x) \quad \text{for } j=1, 2, \dots, 2^n.$$

Consider $\sum_{j=1}^{2^n} M_j \frac{1}{2^n}$, since on $[\frac{1}{n}, 1]$, there are

exact n points, $(1, \frac{1}{2}, \dots, \frac{1}{n})$ have positive value 1,

and $f(x)=0 \quad \forall x \in [\frac{1}{n}, 1] \setminus \{1, \frac{1}{2}, \dots, \frac{1}{n}\}$. Hence

$M_j=1$ if $[\frac{1}{n} + \frac{j-1}{2^n}(1 - \frac{1}{n}), \frac{1}{n} + \frac{j}{2^n}(1 - \frac{1}{n})]$ contains $\frac{1}{k}$

for some $1 \leq k \leq n$, $M_j=0$ otherwise. The number of j such that $M_j=1$ is at most $2n$.

$$\left(\sup_{x \in [0, \frac{1}{n}]} f(x) \leq 1 \right)$$

$$\therefore U(f, P_n) \leq \frac{1}{n} + 2n \cdot \frac{1}{2^n} \quad \text{if } f \geq 0.$$

$$(U) \int_0^1 f(x) dx \geq (L) \int_0^1 f(x) dx \geq L(f, P_n) \geq 0$$

$$\text{But } \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{2n}{2^n} \right) = 0$$

$$\Rightarrow (U) \int_0^1 f(x) dx = (L) \int_0^1 f(x) dx = 0$$

i.e. f is integrable on $[0, 1]$ and $\int_0^1 f(x) dx = 0$

Ex. 1

#4. (a) f is conti. at $x_0 \in [a, b]$ and $f(x_0) \neq 0$.

For $\frac{|f(x_0)|}{2} > 0$, $\exists \delta > 0$ s.t.

$$|f(x) - f(x_0)| < \frac{|f(x_0)|}{2} \quad \text{if } |x - x_0| < \delta \text{ and } x \in [a, b]$$

∴

$$||f(x)| - |f(x_0)||$$

$$\Rightarrow \frac{|f(x_0)|}{2} < |f(x)| \quad \text{if } |x - x_0| < \delta \text{ and } x \in [a, b].$$

Choose c, d s.t. $a \leq c < d \leq b$, $x_0 - \delta < c \leq x_0$, $x_0 \leq d < x_0 + \delta$,

$$\text{then } \forall x \in [c, d], \quad |f(x)| > \frac{|f(x_0)|}{2} \Rightarrow \inf_{x \in [c, d]} |f(x)| \geq \frac{|f(x_0)|}{2}$$

Consider the partition $P = \{a, c, d, b\}$ of $[a, b]$, then

$$\begin{aligned} (\text{L}) \int_a^b |f(x)| dx &\geq L(f, P) \\ &\geq \left(\inf_{x \in [c, d]} f(x) \right) (d - c) \\ &\geq \frac{|f(x_0)|}{2} (d - c) > 0 \end{aligned}$$

(b) (\Leftarrow) Obvious.

(\Rightarrow). f is continuous on $[0, 1] \Rightarrow |f|$ is conti. on $[0, 1]$

$\Rightarrow f$ is integrable on $[0, 1]$ and

$$0 = \int_a^b |f(x)| dx = (\text{L}) \int_a^b |f(x)| dx.$$

But by (a), if there exists $x \in [a, b]$ s.t. $f(x) \neq 0$,

then $(\text{L}) \int_a^b |f(x)| dx > 0 \star$.

$\therefore f(x) = 0 \quad \forall x \in [a, b]$

(c) No. $[a, b] = [-1, 1]$, $f(x) = x$ for $x \in [-1, 1]$.

f is conti. on $[0, 1]$, $f(x) \neq 0$ except $x=0$.

But $\int_{-1}^1 f(x) dx = \int_{-1}^1 x dx = 0$.
↑
(Check it).

§5.1

#8. (a). Since f is increasing on $[a, b]$,

$$M_j(f) = f(x_j), \quad m_j(f) = f(x_{j-1}).$$

$$\begin{aligned} & \sum_{j=1}^n (M_j(f) - m_j(f))(x_j - x_{j-1}) \\ & \leq \sum_{j=1}^n (f(x_j) - f(x_{j-1})) \|P\| \\ & = \|P\| (f(x_n) - f(x_0)) = \|P\| (f(b) - f(a)). \end{aligned}$$

(b). First consider the case that f is increasing on $[a, b]$.

Given $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t. $\frac{f(b) - f(a)(b-a)}{n_0} < \varepsilon$

Let $P = \left\{ a + \frac{j}{n_0}(b-a) \mid j=0, 1, \dots, n_0 \right\}$, a partition of $[a, b]$

By (a),

$$\begin{aligned} U(f, P) - L(f, P) & \leq (f(b) - f(a)) \|P\| \\ & = (f(b) - f(a)) \frac{(b-a)}{n_0} < \varepsilon. \end{aligned}$$

$\therefore f$ is integrable on $[a, b]$.

§5.2

#3. f is integrable on $[0, 1] \Rightarrow f$ is bounded on $[0, 1]$.
 i.e. $\exists M > 0$ s.t. $|f(x)| \leq M \quad \forall x \in [0, 1]$.

$$\begin{aligned} 0 &\leq \left| n^\alpha \int_0^{\frac{1}{n^\alpha}} f(x) dx \right| \leq n^\alpha \int_0^{\frac{1}{n^\alpha}} |f(x)| dx \leq n^\alpha \int_0^{\frac{1}{n^\alpha}} M dx \\ &= M \frac{n^\alpha}{n^\alpha} = \frac{M}{n^{1-\alpha}} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

$$\text{Hence } \lim_{n \rightarrow \infty} n^\alpha \int_0^{\frac{1}{n^\alpha}} f(x) dx = 0.$$

#5(a). f is integrable on $[a, b]$

$$\Rightarrow \exists M > 0 \text{ s.t. } |f(x)| \leq M \quad \forall x \in [a, b].$$

$$\therefore -M g_n(x) \leq f(x) g_n(x) \leq M g_n(x) \quad \forall x \in [a, b] \quad (\because g_n \geq 0)$$

$$\Rightarrow -M \int_a^b g_n(x) dx \leq \int_a^b f(x) g_n(x) dx \leq M \int_a^b g_n(x) dx$$

$$\text{But } \lim_{n \rightarrow \infty} (-M \int_a^b g_n(x) dx) = \lim_{n \rightarrow \infty} (M \int_a^b g_n(x) dx) = 0$$

$$\therefore \lim_{n \rightarrow \infty} \int_a^b f(x) g_n(x) dx = 0.$$

(b). Let $g_n(x) = x^n$. g_n is continuous on $[0, 1]$, hence is integrable on $[0, 1]$. And $g_n \geq 0$ on $[0, 1]$.

$$\int_0^1 x^n dx = \frac{1}{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\text{By (a), } \lim_{n \rightarrow \infty} \int_0^1 f(x) g_n(x) dx = 0.$$

§5.2

#8 (a). $|f(x)| \leq M \quad \forall x \in [a, b]$.

$$\therefore \int_a^b |f(x)|^p dx \leq \int_a^b M^p dx = M^p(b-a).$$

f is conti. on $[a, b] \Rightarrow \exists x_0 \in [a, b]$ s.t.

$$|f(x_0)| = \sup_{x \in [a, b]} |f(x)| = M.$$

f is conti. at x_0 .

\therefore For every $\varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|f(x) - f(x_0)| < \varepsilon \quad \text{if} \quad |x - x_0| < \delta, \quad x \in [a, b]$$

$$|f(x)| > |f(x_0)| - \varepsilon = M - \varepsilon.$$

$$\text{if } x \in I = [c, d] \equiv [a, b] \cap [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}] \subset [a, b]$$

$$\int_a^b |f(x)|^p dx \geq \int_c^d |f(x)|^p dx \geq \int_c^d (M - \varepsilon)^p dx$$

\uparrow
 $\because |f(x)|^p \geq 0$

$$= (M - \varepsilon)^p (d - c) = (M - \varepsilon)^p |I|.$$

(b). By (a),

$$(M - \varepsilon) |I|^{\frac{1}{p}} \leq \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \leq M (b-a)^{\frac{1}{p}}$$

$$\therefore M = \limsup_{p \rightarrow \infty} (M(b-a)^{\frac{1}{p}}) \geq \limsup_{p \rightarrow \infty} \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}$$

$$\geq \liminf_{p \rightarrow \infty} \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \geq \liminf_{p \rightarrow \infty} \left((M - \varepsilon) |I|^{\frac{1}{p}} \right) = M - \varepsilon$$

$$\varepsilon \text{ arbitrary} \Rightarrow M \geq \limsup_{p \rightarrow \infty} \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \geq \liminf_{p \rightarrow \infty} \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \geq M$$

$$\therefore M = \limsup_{p \rightarrow \infty} \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} = \liminf_{p \rightarrow \infty} \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}$$

$\Rightarrow \lim_{p \rightarrow \infty} \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}$ exists and equals to M .

#7. f is diff. on $[a, b] \Rightarrow f$ is conti. on $[a, b]$.

Moreover, f is 1-1 on $[a, b] \Rightarrow f$ is strictly monotone on $[a, b]$ and f^{-1} exists and is conti. on $f([a, b])$.

Case: f is strictly increasing. ($\Leftrightarrow f^{-1}$ is strictly increasing)

$$f([a, b]) = [f(a), f(b)]$$

For each $t \in [a, b]$,

f is integrable, and conti. on $[a, b]$

f^{-1} is integrable and conti. on $[f(a), f(b)]$

Let

$$F(t) = \int_a^t f(x) dx + \int_{f(a)}^{f(t)} f^{-1}(x) dx - tf(t) + af(a)$$

By Thm 5.28, F is differentiable on $[a, b]$

$$F'(t) = f(t) + \underbrace{f^{-1}(f(t)) f'(t)}_{(\text{chain rule})} - f(t) + tf'(t)$$

$$= f(t) + tf'(t) - f(t) - tf'(t) = 0.$$

$$\Rightarrow F(t) \equiv F(a) = 0 \quad \forall t \in [a, b].$$

In particular,

$$0 = F(b) = \int_a^b f(x) dx + \int_{f(a)}^{f(b)} f^{-1}(x) dx - bf(b) + af(a)$$

$$\text{i.e. } \int_a^b f(x) dx + \int_{f(a)}^{f(b)} f^{-1}(x) dx = bf(b) - af(a)$$

Case: f is strictly decreasing

Similar to the previous case, just by letting

$$F(t) = \int_a^t f(x) dx - \int_{f(t)}^{f(a)} f^{-1}(x) dx - tf(t) + af(a).$$

§5.3

#10. (a) f' is conti. on $[a, b] \Rightarrow f'$ is integrable on $[a, b]$

And f, g are both diff. on $[a, b]$, g' is integrable on $[a, b]$

$$\begin{aligned} \Rightarrow \int_a^b f(x)g'(x)dx &= f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx \\ &\quad (\text{by Thm 5.31 [Integration by Parts]}) \\ &= - \int_a^b f'(x)g(x)dx. \end{aligned}$$

(\Leftarrow) f is constant on $[a, b]$

$$\Rightarrow f' = 0 \text{ on } [a, b]$$

$$\Rightarrow \int_a^b f(x)g'(x)dx = - \int_a^b 0 \cdot g(x)dx = 0.$$

$$(\Rightarrow) \int_a^b f'(x)g(x)dx = - \int_a^b f(x)g'(x)dx = 0$$

f is increasing $\Rightarrow f' \geq 0$

$\therefore f'g \geq 0$ on $[a, b]$ ($\because g > 0$ on (a, b) , $g(a)=g(b)=0$)

f', g are conti. on $[a, b] \Rightarrow fg$ is conti. on $[a, b]$

($\because g$ is diff. on $[a, b]$)

\therefore By §5.1 Ex 4.

$$\int_a^b |f'g| dx = \int_a^b f'g dx = 0$$

$$\Rightarrow f'g = 0 \text{ on } [a, b]$$

$g(x) > 0$ on (a, b) $\therefore f'(x) = 0$ on (a, b)

$\Rightarrow f$ is constant on $[a, b]$.

(b). Let $[a, b] = [0, 6]$.

$$f(x) = \begin{cases} -(x-1)^2 & \text{if } x \in [0, 1] \\ 0 & \text{if } x \in [1, 5] \\ (x-5)^2 & \text{if } x \in [5, 6] \end{cases} \quad g(x) = \begin{cases} 0 & \text{if } x \in [0, 2] \cup [4, 6] \\ 1 + \sin\left(\frac{2x-1}{2}\pi\right) & \text{if } x \in [2, 4] \end{cases}$$

f, g satisfies the condition of (a), but $fg' \equiv 0$ on $[0, 6]$

$$\therefore \int_a^b f(x)g'(x)dx = 0$$

35.3

III. ϕ' is conti. on $[a, b]$.

$\phi'(x) \neq 0 \quad \forall x \in [a, b]$

$\Rightarrow \phi'(x) > 0 \quad \forall x \in [a, b] \text{ or } \phi'(x) < 0 \quad \forall x \in [a, b]$

(Intermediate Value Thm)

$\Rightarrow \phi$ is strictly monotone on $[a, b]$.

First assume that ϕ is strictly increasing.

Then $d = \phi(b)$, $c = \phi(a)$, and by Thm 5.34

$$\int_{\phi(a)}^{\phi(b)} f(t) dt = \int_a^b f(\phi(x)) \phi'(x) dx$$

$$\int_c^d f(t) dt = \int_a^b f(\phi(x)) |\phi'(x)| dx$$

Now consider the case ϕ is strictly decreasing

$$\begin{aligned} d &= \phi(a) \\ c &= \phi(b) \end{aligned}$$

Let $\psi = c+d-\phi$. ψ is strictly increasing.

$$\psi([a, b]) = [\psi(a), \psi(b)] = [c+d-\phi(a), c+d-\phi(b)] = [c, d]$$

Let $g: [c, d] \rightarrow \mathbb{R}$, $g(t) = f(c+d-t)$.

Claim: g is integrable, and $\int_c^d g(t) dt = \int_c^d f(t) dt$

Assuming the Claim, then by previous result, we have

$$\int_c^d g(t) dt = \int_a^b g(\psi(x)) |\psi'(x)| dx$$

$$\int_c^d f(t) dt = \int_a^b f(\phi(x)) |\phi'(x)| dx$$

pf of Claim: f is integrable on $[c, d] \Rightarrow$

$$\lim_{\|P\| \rightarrow 0} \sum_{j=1}^n f(t_j)(x_j - x_{j-1}) = \int_c^d f(t) dt.$$

\therefore For any partition $P = \{c = x_0 < x_1 < \dots < x_n = d\}$ of $[c, d]$.

$\tilde{P} = \{y_j \mid y_j = c+d-x_j\}$ is still a partition of $[c, d]$.

$$\xi_j \in [x_{j-1}, x_j] \quad t_j = c+d-\xi_j \in [y_{j-1}, y_j]$$

$$\sum_{j=1}^n g(\xi_j)(x_j - x_{j-1}) = \sum_{j=1}^n f(t_j)(y_j - y_{j-1}) \rightarrow \int_c^d f(t) dt \text{ as } \|P\| \rightarrow 0$$

$\therefore g$ is integrable on $[c, d]$ and $\int_c^d g(t) dt = \int_c^d f(t) dt$