§2.4

#4 Let $S_n = \sum_{k=1}^{n} x_k$. Then the assumption is equivalent to that $\{S_n\}$ is a Cauchy seq. Hence it converges, i.e. $\lim_{n\to\infty} \sum_{k=1}^{n} x_k$ exists and is finite.

#8 (a). Let I be a bdd. closed interval, say $I = [a, b]$.

If $\{x_n\}$ is a seq. in I, then $\{x_n\}$ is bdd.

By Bolzano-Weierstrass Thm, $\exists \{x_{n_k}\}$ converges to $x \in \mathbb{R}$

$$a \le x_{n_k} \le b$$

$$\Rightarrow \quad a \le x = \lim_{k\to\infty} x_{n_k} \le b \qquad \text{i.e. } x \in [a, b]$$

Hence I is sequentially compact.

(b) Example: $(0, 1)$ is a bdd. interval. Let $x_n = \frac{1}{n}$.

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

Hence any subseq. of $\{x_n\}$ converges to 0.

But $0 \notin (0,1)$. $\therefore$ There are no subseqs converging with limit belonging to $(0, 1)$.

$\Rightarrow (0, 1)$ is not sequentially compact.

(c) Example: $[0, \infty)$ is a closed interval. Let $x_n = n$ for each $n \in \mathbb{N}$.

For any subseq. of $\{x_n\}$, it's not bdd., therefore it can't converge. $\therefore \{x_n\}$ has no convergent subseq's.

$\Rightarrow [0, \infty)$ is not sequentially compact.

§2.5

#7. In the case $\limsup_{n\to\infty} x_n$ is finite, the conclusion is true just by the fact that a decreasing convergent seq. converges to the infimum of the seq. as stated in the proof of Thm 2.19 [Monotone Convergence Theorem].

($\because$ $\limsup_{n\to\infty} x_n$ is finite $\Rightarrow$ $\sup_{k\geq n} x_k$ is finite for n large enough. Since $\sup_{k\geq n} x_k$ is decreasing, we can apply the result of Thm 2.19).

Now we just have to prove the case $\limsup_{n\to\infty} x_n = \infty$ or $\limsup_{n\to\infty} x_n = -\infty$.

<u>Case</u>: $\limsup_{n\to\infty} x_n = \infty$.

For any $M>0$, $\exists\, N\in\mathbb{N}$ s.t. $\sup_{k\geq n} x_k \geq M$ if $n\geq N$.

$\Rightarrow$ $\sup_{k\geq n} x_k \geq \sup_{k\geq N} x_k \geq M \quad \forall\, n<N$

$\therefore$ $\sup_{k\geq} x_k \geq M \quad \forall\, n\in\mathbb{N}$

$\Rightarrow$ $\inf_{n\in\mathbb{N}} (\sup_{k\geq n} x_k) \geq M$

M arbitrary. $\Rightarrow$ $\inf_{n\in\mathbb{N}} (\sup_{k\geq n} x_k) = \infty = \limsup_{n\to\infty} x_n$.

<u>Case</u>: $\limsup_{n\to\infty} x_n = -\infty$

For any $M>0$, $\exists\, N\in\mathbb{N}$ s.t. $\sup_{k\geq n} x_k \leq -M$ if $n\geq N$

$\Rightarrow$ $\inf_{n\in\mathbb{N}} (\sup_{k\geq n} x_k) \leq \inf_{n\geq N} (\sup_{k\geq n} x_k) \leq -M$

M arbitrary $\Rightarrow$ $\inf_{n\in\mathbb{N}} (\sup_{k\geq n} x_k) = -\infty = \limsup_{n\to\infty} x_n$

$\therefore$ $\limsup_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} (\sup_{k\geq n} x_k)$.

Similarly $\liminf_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} (\inf_{k\geq n} x_k)$.

§3.1

#8. (a). Given $\varepsilon > 0$, $\because L = \lim_{x\to a} f(x)$

$\therefore \exists\, \delta > 0 \ni |f(x) - L| < \varepsilon$ if $0 < |x-a| < \delta$.

$\therefore \big||f(x)| - |L|\big| \leq |f(x) - L| < \varepsilon$ if $0 < |x-a| < \delta$

(triangle ineq.) (Thm 1.7 (iii))

i.e. $\lim_{x\to a} |f(x)| = |L|$.

(b). Define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Then obviously $\lim_{x\to 0+} f(x) = 1 \neq -1 = \lim_{x\to 0-} f(x)$

$\therefore$ The limit of f at 0 doesn't exist by Thm 3.14.

(or we can prove it directly).

But $|f(x)| = 1 \quad \forall x \in \mathbb{R}$

$\Rightarrow \lim_{x\to 0} |f(x)| = 1$

§3.2

#9. ($\Leftarrow$) Obviously.

($\Rightarrow$) For each $x \in [0,1] \setminus \mathbb{Q}$, $\exists$ a seq. $\{x_n\} \subset \mathbb{Q} \cap [0,1] \setminus \{x\}$ converges to x as $n \to \infty$. (For example, let $x_n = \dfrac{[10^n x]}{10^n} \in \mathbb{Q}$)

$$\therefore f(x) = \lim_{n\to\infty} f(x_n) \quad \text{(Thm3.6 [Sequential Characterization of Limits])}$$
$$= \lim_{n\to\infty} 0 = 0.$$
$$\therefore f(x) = 0 \quad \forall x \in [0,1].$$

#10. For $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.
$$L - \varepsilon < f(n+1) - f(n) < L + \varepsilon \quad \text{if } n \geq N.$$

$\therefore$ If $n > N$,
$$\frac{f(n)}{n} = \frac{1}{n}\Big[f(n) - f(n-1) + \cdots + f(N+1) - f(N)\Big] + \frac{f(N)}{n}$$
$$< \frac{1}{n}\big[(L+\varepsilon) + \cdots + (L+\varepsilon)\big] + \frac{f(N)}{n}$$
$$= \frac{n-N}{n}(L+\varepsilon) + \frac{f(N)}{n}$$

$$\frac{f(n)}{n} > \frac{n-N}{n}(L-\varepsilon) + \frac{f(N)}{n}$$

$$\therefore \quad L - \varepsilon = \liminf_{n\to\infty}\left[\frac{n-N}{n}(L-\varepsilon) + \frac{f(N)}{n}\right]$$
$$\leq \liminf_{n\to\infty} \frac{f(n)}{n} \leq \limsup_{n\to\infty} \frac{f(n)}{n}$$
$$\leq \limsup_{n\to\infty}\left[\frac{n-N}{n}(L+\varepsilon) + \frac{f(N)}{n}\right] = L + \varepsilon$$

ε arbitrary $\Rightarrow$ $L \leq \liminf_{n\to\infty} \dfrac{f(n)}{n} \leq \limsup_{n\to\infty} \dfrac{f(n)}{n} \leq L.$

$$\therefore \quad \limsup_{n\to\infty} \frac{f(n)}{n} = \liminf_{n\to\infty} \frac{f(n)}{n} = L$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(n)}{n} = L.$$

§3.3

#5. Define $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}^c \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q}^c \end{cases}$$

Then f, g are not conti. everywhere, and $f+g \equiv 1$, $fg \equiv 0$ are conti. fcns.

#6. (a). For any $x_0 \in \mathbb{R} \setminus \{0\}$,

$\frac{1}{x}$ is conti. at x_0

and $\cos$ is conti. at $\frac{1}{x_0}$

$\Rightarrow$ Given $\varepsilon > 0$, $\exists\ \hat{\delta} > 0$ s.t.

$$\left| \cos y - \cos \frac{1}{x_0} \right| < \varepsilon \quad \text{if } \left| y - \frac{1}{x_0} \right| < \hat{\delta}$$

For $\hat{\delta} > 0$, $\exists\ \delta > 0$ s.t.

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| < \hat{\delta} \quad \text{if } |x - x_0| < \delta.$$

Therefore if $|x - x_0| < \delta$, then

$$\left| \cos \frac{1}{x} - \cos \frac{1}{x_0} \right| < \varepsilon.$$

i.e. $\lim_{x \to x_0} f(x) = f(x_0)$.

f is conti. at $x_0 \in \mathbb{R} \setminus \{0\}$.

$\frac{1}{2n\pi} \longrightarrow 0$ as $n \longrightarrow \infty$, but $f(\frac{1}{2n\pi}) = \cos 2n\pi = 1 \longrightarrow 1$ $\neq f(0)$ as $n \longrightarrow \infty$. $\therefore$ f is not conti. at 0.

#6(b) (i) f is conti. on $(0, \frac{2}{\pi})$ & g is conti. on $(0, \frac{2}{\pi})$

$\Rightarrow fg$ is conti. on $(0, \frac{2}{\pi})$

(ii) $f(\frac{2}{\pi})g(\frac{2}{\pi}) = \cos\frac{\pi}{2}\, g(\frac{2}{\pi}) = 0$.

f is conti. at $\frac{2}{\pi}$.

$\therefore$ Given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$$|f(x) - f(\tfrac{2}{\pi})| = |f(x)| < \frac{\varepsilon}{C} \quad \text{if } \tfrac{2}{\pi} - \delta < x < \tfrac{2}{\pi} + \delta$$

Note that $|g(x)| \le C\sqrt{x} \le C\sqrt{\frac{2}{\pi}} \le C \quad \forall x \in (0, \frac{2}{\pi})$

$$\therefore \quad |f(x)g(x)| \le \frac{\varepsilon}{C} \cdot C\sqrt{x}$$
$$= \varepsilon\sqrt{x}$$
$$\le \varepsilon \quad \text{if } \tfrac{2}{\pi} - < x < \tfrac{2}{\pi}$$

$\therefore \lim_{x \to \frac{2}{\pi}-} f(x)g(x) = 0 = f(\frac{2}{\pi})g(\frac{2}{\pi})$

$\therefore f$ is conti. at $\frac{2}{\pi}$.

(iii) $f(0)g(0) = 0 \cdot g(0) = 0$.

Given $\varepsilon > 0$, take $\delta = \frac{\varepsilon^2}{4C^2} > 0$,

if $0 < x < \delta$, then

$$|f(x)g(x)| \le |\cos\tfrac{1}{x}|\, C\sqrt{x}$$
$$\le C\sqrt{x}$$
$$\le C\sqrt{\frac{\varepsilon^2}{4C^2}} = \frac{\varepsilon}{2} < \varepsilon.$$

$\therefore \lim_{x \to 0+} f(x)g(x) = 0 = f(0)g(0)$.

$\therefore f$ is conti. at 0.

Combine (i), (ii), (iii), fg is conti. on $[0, \frac{2}{\pi}]$

§3.3

#10. $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$.

$\therefore$ For $|f(0)|+1>0$, $\exists M_1, M_2 > 0$ s.t.

$$f(x) > |f(0)|+1 \quad \text{if} \quad x > M_1$$

$$f(x) > |f(0)|+1 \quad \text{if} \quad x < -M_2.$$

Let $I = [-M_2, M_1]$.

$f\restriction_I : I \longrightarrow \mathbb{R}$ is conti.

$\Rightarrow \exists\, x_m \in I = [-M_2, M_1]$ s.t.

$$f(x_m) = \inf_{x\in I} f(x)$$

Note that $f(x_m) = \inf_{x\in I} f(x) \leq f(0)$

$$< |f(0)|+1 < f(x) \quad \forall x \in (-\infty, -M_2) \cup (M_1, \infty).$$

And $f(x_m) = \inf_{x\in I} f(x) \leq f(x) \quad \forall x \in I = [-M_2, M_1]$

$\therefore \quad f(x_m) \leq f(x) \quad \forall x \in \mathbb{R}$

$$\Rightarrow \quad \underset{\displaystyle \inf_{x\in\mathbb{R}} f(x)}{\underset{\rotatebox{90}{$\leq$}}{f(x_m)}} \leq \inf_{x\in\mathbb{R}} f(x) \leq f(0) < \infty$$

$$\Rightarrow \quad f(x_m) = \inf_{x\in\mathbb{R}} f(x) < \infty$$

§3.4

#3.
$$\lim_{x\to 0+} x^\alpha \sin\frac{1}{x} \begin{cases} = 0 & \text{if } \alpha>0 \\ \text{does not exist} & \text{if } \alpha\le 0 \end{cases}$$

$$\lim_{x\to 1-} x^\alpha \sin\frac{1}{x} = \sin 1$$

$\therefore$ $x^\alpha \sin\frac{1}{x}$ can be continuously extended to $[0, \frac{2}{\pi}]$ only if $\alpha>0$. By Thm 3.40, $x^\alpha\sin\frac{1}{x}$ is uniformly conti. only if $\alpha>0$.

#4 (a). Given $\varepsilon>0$, $\exists M>0$ s.t.

$$|f(x)-L|<\frac{\varepsilon}{2} \quad \text{if} \quad x>M.$$

$f|_{[0,M+1]} : [0,M+1]\longrightarrow \mathbb{R}$ is conti, hence is uniformly conti. $\Rightarrow \exists\ 1>\delta>0$ s.t.

$$|f(x)-f(y)|<\varepsilon \quad \text{if} \quad |x-y|<\delta \text{ and } x,y\in[0,M+1]$$

Then if $|x-y|<\delta$, $x,y\in[0,\infty)$,

Case1: at least one of x,y lies in $(M+1,\infty)$

$\Rightarrow$ both of x,y lies in (M,∞)

$(\because\ |x-y|<\delta<1)$

$$\therefore |f(x)-f(y)|\le |f(x)-L|+|L-f(y)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Case2: both x,y lies in $[0,M+1]$

$|x-y|<\delta$ and $x,y\in[0,M+1]$

$\Rightarrow |f(x)-f(y)|<\varepsilon.$

$\therefore |f(x)-f(y)|<\varepsilon$ if $|x-y|<\delta$, $x,y\in[0,\infty)$

$\therefore$ f is uniformly conti on $[0,\infty)$.

§3.4

#4. (b) $\lim_{x\to\infty} f(x) = \lim_{x\to -\infty} f(x) = 0$.

Given $\varepsilon > 0$, $\exists$ $M > 0$ s.t

$$|f(x)| < \frac{\varepsilon}{2} \text{ if } |x| > M.$$

$f|_{[-M-1, M+1]} : [-M-1, M+1] \longrightarrow \mathbb{R}$ is conti.

$\Rightarrow$ $f|_{[-M-1,M+1]}$ is uniformly conti.

$\Rightarrow$ $\exists$ $1 > \delta > 0$. s.t.

$$|f(x) - f(y)| < \varepsilon \quad \text{if} \quad |x-y| < \delta, \; x, y \in [-M-1, M+1].$$

$\therefore$ If $|x-y| < \delta$, $x, y \in \mathbb{R}$

<u>Case 1</u>: One of x, y lies in $(-\infty, -M-1) \cup (M+1, \infty)$

$\Rightarrow$ Both x, y lie in $(-\infty, -M) \cup (M, \infty)$

$(\because |x-y| < \delta < 1)$

$$\therefore |f(x) - f(y)| \le |f(x)| + |f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

<u>Case 2</u>: Both x, y lie in $[-M-1, M+1]$

$|x-y| < \delta$ & $x, y \in [-M-1, M+1]$

$\Rightarrow |f(x) - f(y)| < \varepsilon$.

$\therefore$ $|f(x) - f(y)| < \varepsilon$ if $|x-y| < \delta$.

i.e. f is uniformly continuous.

§3.4

#7. (a). Consider $f(a,b)$.

$(a,b) \neq \phi \Rightarrow f(a,b) \neq \phi$

f is bdd $\Rightarrow f(a,b)$ is bdd.

$\therefore f(a,b)$ has finite supremum and finite infimum.

Claim: $f(a+) = \inf f(a,b) \triangleq l$, $f(b-) = \sup f(a,b) \triangleq m$

Given $\varepsilon > 0$, $\exists\ c, d \in (a,b)$ s.t.

$$l \le f(c) < l + \varepsilon$$

$$m - \varepsilon < f(d) \le m$$

Let $\delta_1 = c - a > 0$, $\delta_2 = b - d > 0$,

then if $a < x < a + \delta_1 = c$, $d = b - \delta_2 < y < b$

$$l \le \inf f(a,b) \le f(x) \le f(c) < l + \varepsilon$$

$$m - \varepsilon < f(d) \le f(y) \le \sup f(a,b) = m$$

$\therefore f(a+) = l$, $f(b-) = m$.

(b). ($\Leftarrow$) Obviously.

($\Rightarrow$) By (a), f can be extended to $[a,b]$ to be continuous $\Rightarrow f$ is uniformly conti. by Thm 3.40.

(c). Example: $g : (0,1) \longrightarrow \mathbb{R}$

$$g(x) = \frac{1}{1-x}.$$

§3.4

#8 f is conti. on $[0,1] \Rightarrow f$ is uniformly conti.

Given $\varepsilon>0$, $\exists\, \delta>0$ s.t.

$$|f(x)-f(y)|<\frac{\varepsilon}{2} \quad \text{if} \quad |x-y|<\delta, \quad x,y\in[0,1].$$

$\exists N\in\mathbb{N}$ s.t.

$$\frac{1}{2^n}<\delta \qquad \text{whenever } n\geq N.$$

For $k\in\mathbb{N}$, $k\leq 2^n$,

fix $y\in I_k$, for any $x\in I_k$

$$|x-y|\leq\frac{1}{2^n}<\delta$$

$$\Rightarrow |f(x)-f(y)|<\frac{\varepsilon}{2}$$

$$\therefore\ f(x)<f(y)+\frac{\varepsilon}{2} \qquad \forall x\in I_k$$

$$\Rightarrow \sup_{x\in I_k} f(x)\leq f(y)+\frac{\varepsilon}{2}$$

$$\sup_{x\in I_k} f(x)\leq f(y)+\frac{\varepsilon}{2} \qquad \forall y\in I_k$$

$$\Rightarrow \sup_{x\in I_k} f(x) \leq \inf_{y\in I_k} f(y)+\frac{\varepsilon}{2}$$

$$\Rightarrow \sup_{x\in I_k} f(x)-\inf_{x\in I_k} f(x)\leq\frac{\varepsilon}{2}<\varepsilon.$$