HW §1.4

2. Case \( a = \phi \) : \( B = f(A) = \phi \). \( \therefore \) \( B \) is finite.

Case \( a \neq \phi \) : Since \( A \neq \phi \), \( A \) is finite \( \Leftrightarrow \exists n \in \mathbb{N} \) and a 1-1 function \( g \) from \( \{1, 2, 3, \ldots, n\} \) onto \( A \). Then \( f \circ g \) is a 1-1 function from \( \{1, 2, 3, \ldots, n\} \) onto \( B \). (\( f \circ g \) is 1-1 and onto by §1.4 Ex9(a).) Therefore, \( B \) is finite.

4. (a) \( f(E) = (-4, 16) \),
   \[ f^{-1}(E) = \left(0, \frac{4}{3}\right) \]

(b) \( f(E) = [0, 16] \),
   \[ f^{-1}(E) = [-2, 2] \]

(c) \( f(E) = [-\frac{1}{3}, 2] \),
   \[ f^{-1}(E) = \left(-\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right) \]

(d) \( f(E) = \ln \frac{7}{1}, \ln 31 \),
   \[ f^{-1}(E) = \left[-\frac{1-\sqrt{4e^2-3}}{2}, -\frac{1-\sqrt{4e^2-3}}{2}\right] \cup \left(-\frac{1+\sqrt{4e^2-3}}{2}, -\frac{1+\sqrt{4e^2-3}}{2}\right) \]

(e) \( f(E) = [-1, 1] \),
   \[ f^{-1}(E) = \cup_{n \in \mathbb{N}} [2n\pi, (2n + 1)\pi] \]

6. (a) \( \Rightarrow \) (b) :
   If \( y \in f(A) \setminus f(B) \), then \( y \notin f(A) \) and \( y \notin f(B) \). Therefore,
   \[ y = f(x_1) \text{ for some } x_1 \in A \]
   \[ y \notin f(x) \forall x \in B \]
   \[ \Rightarrow y = f(x_1) \in f(A \setminus B) \]
   \[ \therefore f(A \setminus B) \subset f(A) \]
   Suppose that \( y \in f(B) \), then \( y = f(x_2) \) for some \( x_2 \in B \). But \( f \) is 1-1 and
   \[ y \notin f(B) \]
   \[ \therefore y \notin f(A \setminus f(B), f(A \setminus B) \subset f(A) \setminus f(B) \]
   \[ \therefore f(A \setminus B) = f(A) \setminus f(B) \] for all subsets \( A \) and \( B \) of \( X \).

(b) \( \Rightarrow \) (c) :
   For each \( x \in E, f(x) \in f(E) \Rightarrow x \in f^{-1}(f(E)) \)
   \[ \therefore E \subset f^{-1}(f(E)) \]
   For each \( x \in f^{-1}(f(E)) \), \( f(x) \in f(E) \). Then by (b),
   \[ f(E \setminus \{x\}) = f(E \setminus \{f(x)\}) \]
   \[ \therefore E \setminus \{x\} \subset f(E) \]
   \[ \therefore f^{-1}(f(E)) = E \]

(c) \( \Rightarrow \) (d) :
   For \( y \in f(A) \), \( y = f(x) \) for some \( x \in A \).
   \[ \therefore x \in A \]
   \[ \therefore x \in B \]
   \[ \therefore y = f(x) \in f(B) \]
   \[ \therefore f(A \cap B) \subset f(A) \cap f(B) \]
   For \( y \in f(A) \cap f(B) \),
   \[ y \in f(A) \Rightarrow \exists x \in A \exists y = f(x) 
   y = f(x) \in f(B) \Rightarrow x \in f^{-1}(f(B)) = B \text{ by (c)} \]
   \[ \therefore x \in A \cap B \Rightarrow y \in f(x) \in f(A \cap B) \]
   \[ \therefore f(A \cap B) \subset f(A \cap B) \]
   Hence \( f(A \cap B) = f(A) \cap f(B) \).
(d) \(\Rightarrow\) (a) :
Suppose \(f(x_1) = f(x_2)\) for some \(x_1, x_2 \in X\). Apply (d) by \(A = \{x_1\}, B = \{x_2\}\), we get
\[
\begin{align*}
  f(A) \cap f(B) &= \{f(x_1)\} \cap \{f(x_2)\} = \{f(x_1)\} \neq \emptyset \\
  f(A \cap B) &= f(\{x_1\} \cap \{x_2\}) \\
  \therefore \{x_1\} \cap \{x_2\} \neq \emptyset \Rightarrow x_1 = x_2. \\
\end{align*}
\]

9. (a) (i) Suppose \((g \circ f)(x_1) = (g \circ f)(x_2)\) for some \(x_1, x_2 \in A\) i.e. \(g(f(x_1)) = g(f(x_2))\). Then
\[
\begin{align*}
  g : B \to C &\text{ is 1-1} \quad \therefore f(x_1) = f(x_2) \\
  f : A \to B &\text{ is 1-1} \quad \therefore x_1 = x_2 \\
  \therefore g \circ f &\text{ is 1-1 if } g, f \text{ are 1-1.}
\end{align*}
\]

(ii) Let \(z \in C\).
\[
\begin{align*}
  z \in C \text{ and } g : B \to C &\text{ is onto} \quad \therefore \exists y \in B \ni z = g(y) \\
  y \in B \text{ and } f : A \to B &\text{ is onto} \quad \therefore \exists x \in A \ni y = f(x) \\
  \therefore z = g(f(x)) = (g \circ f)(x) &\text{ for } x \in A. \Rightarrow g \circ f \text{ is onto.}
\end{align*}
\]

(b) Given \(y \in f(A)\), there is \(x \in A\) such that \(y = f(x)\). If there is another \(\tilde{x} \in A\) such that \(y = f(\tilde{x})\), then
\[
y = f(x) = f(\tilde{x}) \quad \text{and } \quad f \text{ is 1-1 } \Rightarrow x = \tilde{x}
\]

Therefore we can define \(f^{-1} : f(A) \to A\) by the way that \(f^{-1}\) maps \(y \in f(A)\) to the unique \(x \in A\) which has \(y\) as its image. By the definition of \(f^{-1}\), if \(f^{-1}(y_1) = f^{-1}(y_2) = x \in A\), then \(y_1 = f(x) = y_2\). \(\therefore f^{-1}\) is 1-1. Now to prove that \(f^{-1}\) is onto. For \(x \in A\), \(f(x) \in f(A)\). By our definition of \(f^{-1}\), we have \(f^{-1}(f(x)) = x\) for \(f(x) \in f(A)\). \(\therefore f^{-1}\) is onto.

(c). \textbf{f is 1-1 }\Leftrightarrow\textbf{ g of is 1-1} :
\[
\begin{align*}
  \Rightarrow &\text{ By (a). } \\
  \Leftarrow &\text{ If } f(x_1) = f(x_2) \text{ for some } x_1, x_2 \in A. \text{ Then } (g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2) \text{ and } g \circ f \text{ is 1-1 } \Rightarrow x_1 = x_2. \therefore f \text{ is 1-1.}
\end{align*}
\]

\textbf{f is onto }\Leftrightarrow\textbf{ g of is onto} :
\[
\begin{align*}
  \Rightarrow &\text{ By (a). } \\
  \Leftarrow &\text{ For each } y \in B, g(y) \in C \text{ and } g \circ f \text{ is onto.} \\
  \Rightarrow &\exists x \in A \ni g(y) = (g \circ f)(x) = g(f(x)) \\
\end{align*}
\]

. Then since \(g\) is 1-1, \(y = f(x)\). \(\therefore\) \(f\) is onto.

11. (a) \(n \in \mathbb{N}, q \in \mathbb{Q}\)

(i) \(q \geq 0\). Then \(q = \frac{p}{r}\) for some \(p \in \mathbb{N}, r \in \mathbb{N} \cup \{0\}\).
\[
(n^q)^p = (n^\frac{p}{r})^p = n^r
\]
\(\therefore\) \(n^q\) is a root of the polynomial \(x^p - n^r\).
( Note that \(p \in \mathbb{N}\) and \(r \in \mathbb{N} \cup \{0\} \Rightarrow n^r \in \mathbb{N}\) )
\(\therefore\) \(n^q\) is algebraic.
(ii) \( q < 0 \). Then \( q = \frac{r}{p} \) for some \( p \in \mathbb{N}, r \in \mathbb{N}\{0\} \).

\[
(n^r)(n^q)^r = (n^r)(n^\frac{r}{p})^r = (n^r)(n^{-r}) = 1. 
\]

\( \therefore n^q \) is a root of the polynomial \( n^r x^{-p} - 1 \)

(Note that \( -p \in \mathbb{N} \) and \( n^r \in \mathbb{N} \))

\( \therefore n^q \) is algebraic.

(b). For each \( n \in \mathbb{N} \), the set \( \mathbb{Z}_n[x] \) of all polynomials with integer coefficients of degree \( n \) is countable. (By considering the choice of coefficients, it is easy to see that there is a 1-1 and onto function from \( \mathbb{Z}_n[x] \) to \( \mathbb{N}^n \). Hence \( P_n[x] \) is countable.) And each of the polynomial has at most \( n \) roots. Hence the set of all roots of polynomials in \( \mathbb{Z}_n[x] \) is countable. The collection of algebraic numbers of degree \( n \) is a subset of the previous set, hence it is countable too.

(c). Let \( A_n = \) the collection of all algebraic numbers of degree \( n \),

\( A = \) the collection of all algebraic numbers,

\( T = \) the collection of all transcendental numbers.

\( \therefore A = \bigcup_{n=1}^{\infty} A_n \) is countable. (Note that a countable union of countable sets is still countable.) Then since \( \mathbb{R} \) is uncountable and \( \mathbb{R} = A \cup T \), \( T \) is uncountable (or \( \mathbb{R} = A \cup T \) is countable, a contradiction).

HW §2.1

4. (a). Given \( \epsilon > 0 \), there is \( N \in \mathbb{N} \) such that \( |b_n - 0| = b_n < \epsilon \) if \( n \geq N \). \( \therefore \) For \( n \geq N \), \( n \) large

\[
|b_n - \epsilon| \leq |b_n - a| \leq \epsilon
\]

i.e. \( \lim_{n \to \infty} x_n = a \).

(b). Let \( \tilde{b}_n = cb_n \), then \( \tilde{b}_n \geq 0 \) and \( \lim_{n \to \infty} \tilde{b}_n = C \lim_{n \to \infty} b_n = C \cdot 0 = 0 \).

And \( |x_n - a| \leq C b_n = \tilde{b}_n \). Hence \( \{\tilde{b}_n\} \) satisfies the same condition as \( \{b_n\} \) in (a).

\( \therefore x_n \) converges to \( a \).

6. (a). \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n \triangleq l \).

Given \( \epsilon > 0 \), there are \( N_1, N_2 \in \mathbb{N} \) such that

\[
|x_n - l| < \frac{\epsilon}{2} \quad \text{if} \quad n \geq N_1
\]

\[
|y_n - l| < \frac{\epsilon}{2} \quad \text{if} \quad n \geq N_2
\]

Let \( N = \max\{N_1, N_2\} \in \mathbb{N} \), then \( n \geq N \) \( \Rightarrow \)

\[
|x_n - y_n| \leq |x_n - l| + |l - y_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad (\because n \geq N \geq N_1 \text{ and } n \geq N \geq N_2)
\]

\( \therefore x_n - y_n \to 0 \) as \( n \to \infty \).
(b). For each $M \in \mathbb{R}$, by Archimedean Principle, there is $n_M \in \mathbb{N}$ such that $n_0 > M$. Therefore, $\{n\}$ is not bounded above. $\Rightarrow \{n\}$ is not bounded. $\Rightarrow \{n\}$ does not converge by Theorem 2.8.

(c). Let $x_n = n$, $y_n = n + \frac{1}{n}$, then $|x_n - y_n| = \frac{1}{n} \to 0$ as $n \to \infty$. But $\{x_n\} = \{n\}$ does not converge by (b). (Note that $\{y_n\}$ does not converge too.)

HW §2.2

5. Let $r_n = \lfloor \frac{10^n x}{10^n} \rfloor \in \mathbb{Q}$. Then $\{r_n\} \to x$ as $n \to \infty$.

10. (a). $0 \leq y < \frac{1}{10^n} \Rightarrow 0 \leq 10^{n+1}y < 10$. Let $A = \{k \in \mathbb{N} : k \leq 10^{n+1}y\}$. Then $0 \in A \neq \phi$ and $\forall k \in A$, $k \leq 10^{n+1}y < 10$ i.e. $A$ has an upper bound 10. $\Rightarrow A$ has a supremum $w$. It is easy to see that $w \in A$. $\Rightarrow 0 \leq w \leq 9$. $\Rightarrow w$ is the supremum of $A$, so $w + 1 \notin A$. $\Rightarrow w + 1 > 10^{n+1}y$. Hence $w \leq 10^{n+1}y < w + 1$. This implies that

$$\frac{w}{10^{n+1}} \leq y \leq \frac{w}{10^{n+1}} + \frac{1}{10^{n+1}}$$

(b). $0 \leq x < \frac{1}{10^n} = 1$. By (a), there is $0 \leq x_1 \leq 9$, $x_1 \in \mathbb{Z}$ such that

$$\frac{x_1}{10} \leq x < \frac{x_1}{10} + \frac{1}{10}$$

$$\Rightarrow 0 \leq x - \frac{x_1}{10} < \frac{1}{10}$$

If there are $x_1, x_2, \ldots, x_n \in \{0, 1, \ldots, 9\}$ such that

$$\sum_{k=1}^{n} \frac{x_k}{10^k} \leq x < \sum_{k=1}^{n} \frac{x_k}{10^k} + \frac{1}{10^n}$$

$$\Rightarrow 0 \leq x - \sum_{k=1}^{n} \frac{x_k}{10^k} < \frac{1}{10^n}$$

By (a), there is $x_{n+1} \in \mathbb{Z}$, $0 \leq x_{n+1} \leq 9$ such that

$$\frac{x_{n+1}}{10^{n+1}} \leq x - \sum_{k=1}^{n} \frac{x_k}{10^k} < \frac{x_{n+1}}{10^{n+1}} + \frac{1}{10^{n+1}}$$

$$\Rightarrow 0 \leq x - \sum_{k=1}^{n+1} \frac{x_k}{10^k} < \frac{1}{10^n}$$

By induction on $n$. q.e.d.

(c). According to (b), $\sum_{k=1}^{n} \frac{x_k}{10^k}$ is a increasing sequence on $n$, bounded above by $x$. $\Rightarrow \sum_{k=1}^{n} \frac{x_k}{10^k}$ converges as $n \to \infty$. Let $n \to \infty$ in the inequality in (b), we get

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{x_k}{10^k} \leq x \leq \lim_{n \to \infty} \sum_{k=1}^{n} \frac{x_k}{10^k} + \lim_{n \to \infty} \frac{1}{10^n} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{x_k}{10^k}$$

i.e. $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{x_k}{10^k} = x$
HW §2.3

4. Note that $x_n \leq 1$ for all $n \in \mathbb{N}$. \(\Rightarrow\) $x_n = 1 - \sqrt{1 - x_{n-1}} \geq 1 - \sqrt{1 - 1} = 0 \quad \forall n \in \mathbb{N}, \ n \geq 2$.

\[ \therefore 0 \leq x_n \leq 1 \quad \forall n \in \mathbb{N} \]

\[ \Rightarrow \quad \sqrt{1 - x_n} = 1 - x_{n+1} \]

\[ \Rightarrow \quad 1 - x_n = 1 - 2x_{n+1} + x_{n+1}^2 \]

\[ \Rightarrow \quad x_n - x_{n+1} = x_{n+1}(1 - x_{n+1}) \geq 0 \]

\[ \therefore x_n \geq x_{n+1} \]

\(\{x_n\}\) decreases and is bounded below by 0. Hence it converges to some $x \in \mathbb{R}$.

Let $n \to \infty$ in the equality $x_{n+1} = 1 - \sqrt{1 - x_n}$, we get

\[ x = 1 - \sqrt{1 - x} \]

\[ \Rightarrow x = 0 \text{ or } 1 \]

But $x_n$ is decreasing, therefore $x \leq x_1 < 1$. \(\therefore\) $x = 0$. i.e. $x_n \downarrow 0$ as $n \to \infty$.

\[ \frac{x_{n+1}}{x_n} = \frac{1 - \sqrt{1 - x_n}}{x_n(1 - x_n)} \]

\[ = \frac{1}{1 + \sqrt{1 - x_n}} \]

\[ \to \frac{1}{1 + \sqrt{0}} = \frac{1}{2} \text{ as } n \to \infty \]

9. (a). For $n = 1$, $x_1 > y_1 > 0$. Suppose that $x_n > y_n > 0$ for some $n \in \mathbb{N}$, then

\[ x_{n+1} = \frac{x_n + y_n}{2} > 0 \]

\[ y_{n+1} = \sqrt{x_ny_n} > 0 \]

\[ x_{n+1}^2 - y_{n+1}^2 = \frac{(x_n - y_n)^2}{4} > 0 \]

\[ \Rightarrow x_{n+1} > y_{n+1} \]

\[ \Rightarrow x_{n+1} > y_{n+1} > 0 \]

By induction on $n$, $0 < y_n < x_n$ for all $n \in \mathbb{N}$.

(b).

\[ x_{n+1} = \frac{x_n + y_n}{2} < \frac{x_n + x_n}{2} = x_n \]

\[ y_{n+1} = \sqrt{x_ny_n} > \sqrt{y_ny_n} = y_n \]

\[ y_1 < y_n < x_n < x_1 \text{ by (a)} \]

Therefore $\{x_n\}$ is strictly decreasing and bounded below by $y_1$ and $\{y_n\}$ is strictly increasing and bounded above by $x_1$. By Monotone Convergence Theorem, the two sequences converge.
(c). \[ 0 < x_2 - y_2 = \frac{x_1 + y_1}{2} - \sqrt{x_1y_1} < \frac{x_1 + y_1}{2} - \frac{\sqrt{y_1y_2}}{2} = \frac{x_1 - y_1}{2} \]

Suppose that \( x_{n+1} - y_{n+1} < \frac{x_1 - y_1}{2n+1} \), then
\[ x_{n+2} - y_{x+2} = \frac{x_{n+1}}{y_{n+1}} - \sqrt{x_{n+1}y_{n+1}} < \frac{x_{n+1}}{y_{n+1}} - \frac{x_{n+1} - y_{n+1}}{2} < \frac{x_1 - y_1}{2n+1} \]

By induction on \( n \), \( x_{n+1} - y_{n+1} < \frac{x_1 - y_1}{2n+1} \) for all \( n \in \mathbb{N} \).

(d). \( \lim_{n \to \infty} \frac{x_1 - y_1}{2n} = 0 \), and \( 0 < x_n - y_n < \frac{x_1 - y_1}{2n} \) for all \( n \in \mathbb{N} \), \( n \geq 2 \).
\[ \Rightarrow 0 = \lim_{n \to \infty} (x_n - y_n) = \lim_{n \to \infty} x_n - \lim_{n \to \infty} y_n = 0. \therefore \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n. \]

11. (a). Claim: \( x_n > y_n > 0 \) for all \( n \in \mathbb{N} \cup \{0\} \).

\[ x_0 = 2\sqrt{3} > 3 = y_0 > 0 \]

Suppose \( x_n > y_n > 0 \) for some \( n \in \mathbb{N} \)
\[ x_{n+1} = \frac{2x_ny_n}{x_n + y_n} > \frac{2x_ny_n}{x_n + x_n} = y_n > 0 \]
\[ \therefore y_{n+1} = \sqrt{x_{n+1}y_{n+1}} < \sqrt{x_{n+1}x_{n+1}} = x_{n+1} \]

By induction on \( n \), the Claim holds for all \( n \in \mathbb{N} \cup \{0\} \)

According to the Claim, for each \( n \in \mathbb{N} \)
\[ x_n = \frac{2x_{n-1}y_{n-1}}{x_{n-1} + y_{n-1}} < \frac{2x_{n-1}y_{n-1}}{y_{n-1} + y_{n-1}} = x_{n-1} \]
\[ y_n = \sqrt{x_ny_{n-1}} > \sqrt{y_ny_{n-1}} \Rightarrow \sqrt{y_n} > \sqrt{y_{n-1}} \Rightarrow y_n > y_{n-1} \]
\[ \Rightarrow x_0 > x_n > y_n > y_0 \]

\( \therefore \{x_n\} \) decreases and is bounded below by \( y_0 \), hence has limit \( x \in \mathbb{R} \); \{y_n\} increases and is bounded above by \( x_0 \), hence has limit \( y \in \mathbb{R} \).

(b).
\[ x_0 > x_n > y_n > y_0 \]
\[ 2\sqrt{3} = x_0 \geq x \geq y \geq y_0 = 3 > 0 \]

Let \( n \to \infty \) in \( x_n = \frac{2x_{n-1}y_{n-1}}{x_{n-1} + y_{n-1}} \) to get \( x = \frac{2xy}{x+y} \).
\[ \Rightarrow x^2 + xy = 2xy \Rightarrow x^2 = xy \Rightarrow x = y \ (\because x > 0) \]