HW §1.4

2. **Case** $a = \phi$: $B = f(A) = \phi$. $\therefore B$ is finite.

   Case $a \neq \phi$: Since $A \neq \phi$, A is finite $\Leftrightarrow \exists\, n \in \mathbb{N}$ and a 1-1 function g from $\{1, 2, 3, \ldots, n\}$ onto A. Then $f \circ g$ is a 1-1 function from $\{1, 2, 3, \ldots, n\}$ onto B. ($f \circ g$ is 1-1 and onto by §1.4 Ex9(a).) Therefore, B is finite.

4. (a). $f\,(E) = (-4, 16)$,
   $f^{-1}(E) = (0, \frac{4}{5})$

   (b). $f\,(E) = [0, 16]$,
   $f^{-1}(E) = [-2, 2]$

   (c). $f\,(E) = [-\frac{1}{4}, 2]$,
   $f^{-1}(E) = (\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2})$

   (d). $f\,(E) = (\ln\frac{7}{4}, \ln 31)$,
   $f^{-1}(E) = [\frac{-1-\sqrt{4e^5-3}}{2}, \frac{-1-\sqrt{4e^{1/2}-3}}{2}) \cup (\frac{-1+\sqrt{4e^{1/2}-3}}{2}, \frac{-1+\sqrt{4e^5-3}}{2}]$

   (e). $f\,(E) = [-1, 1]$,
   $f^{-1}(E) = \cup_{n\in\mathbb{N}}\, [2n\pi, (2n+1)\pi]$

6. **(a) $\Rightarrow$ (b)** :

   If $y \in f(A)\backslash f(B)$, then $y \in f(A)$ and $y \notin f(B)$. Therefore,
   $$\left.\begin{array}{l} y = f(x_1) \text{ for some } x_1 \in A \\ y \neq f(x)\ \forall x \in B \end{array}\right\} \Rightarrow x_1 \notin B \Rightarrow x_1 \in A\backslash B$$
   $\Rightarrow y = f(x_1) \in f(A\backslash B)$. $\quad\therefore f(A)\backslash f(B) \subset f(A\backslash B)$.
   If $y \in f(A\backslash B)$, then $y = f(x_1)$ for some $x_1 \in A\backslash B$. $\Rightarrow y = f(x_1) \in f(A)$
   Suppose that $y \in f(B)$, then $y = f(x_2)$ for some $x_2 \in B$. But f is 1-1 and $y = f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \in B$. This contradicts to that $x_1 \in A\backslash B$. Hence $y \notin f(B)$. $\therefore y \in f(A)\backslash f(B)$. $\Rightarrow f(A\backslash B) \subset f(A)\backslash f(B)$.
   $\therefore f(A\backslash B) = f(A)\backslash f(B)$ for all subsets A and B of X.

   (b) $\Rightarrow$ (c) :

   For each $x \in E$, $f(x) \in f(E) \Rightarrow x \in f^{-1}(f(E))$. $\therefore E \subset f^{-1}(f(E))$.
   For each $x \in f^{-1}(f(E))$, $f(x) \in f(E)$. Then by (b),
   $$f(E\backslash\{x\}) = f(E)\backslash f(\{x\}) = f(E)\backslash\{f(x)\} \subsetneqq f(E)$$
   $\therefore E\backslash\{x\} \subsetneqq E \Rightarrow \{x\} \subset E \Rightarrow x \in E$. $\therefore f^{-1}(f(E)) \subset E$
   $\therefore f^{-1}(f(E)) = E$.

   (c) $\Rightarrow$ (d) :

   For $y \in f(A \cup B)$, $y = f(x)$ for some $x \in A \cap B$.
   $$\left.\begin{array}{ll} x \in A & \therefore y = f(x) \in f(A) \\ x \in B & \therefore y = f(x) \in f(B) \end{array}\right\} \Rightarrow y \in f(A) \cap f(B).$$
   $\therefore f(A \cap B) \subset f(A) \cap f(B)$.
   For $y \in f(A) \cap f(B)$,
   $$\begin{array}{lll} y \in f(A) & \Rightarrow & \exists\, x \in A \ni y = f(x) \\ y = f(x) \in f(B) & \Rightarrow & x \in f^{-1}(f(B)) = B \text{ by } (c) \end{array}$$
   $\therefore x \in A \cap B \Rightarrow y = f(x) \in f(A \cap B)$. $\therefore f(A) \cap f(B) \subset f(A \cap B)$.
   Hence $f(A \cap B) = f(A) \cap f(B)$.

(d) $\Rightarrow$ (a) :

Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$. Apply (d) by $A = \{x_1\}$, $B = \{x_2\}$, we get

$$\begin{array}{rcl} f(A) \cap f(B) & = & \{f(x_1)\} \cap \{f(x_2)\} = \{f(x_1)\} \neq \phi \\ \| & & \\ f(A \cap B) & = & f(\{x_1\} \cap \{x_2\}) \end{array}$$

$\therefore \{x_1\} \cap \{x_2\} \neq \phi \Rightarrow x_1 = x_2$. $\therefore f$ is 1-1 on X.

9. (a). (i) Suppose $(g \circ f)(x_1) = (g \circ f)(x_2)$ for some $x_1, x_2 \in A$. i.e. $g(f(x_1)) = g(f(x_2))$. Then

$$\begin{array}{ll} g : B \to C \text{ is 1-1} & \therefore f(x_1) = f(x_2) \\ f : A \to B \text{ is 1-1} & \therefore x_1 = x_2 \end{array}$$

$\therefore g \circ f$ is 1-1 if g, f are 1-1.

(ii) Let $z \in C$.

$$\begin{array}{ll} z \in C \text{ and } g : B \to C \text{ is onto} & \therefore \exists y \in B \ni z = g(y) \\ y \in B \text{ and } f : A \to B \text{ is onto} & \therefore \exists x \in A \ni y = f(x) \end{array}$$

$\therefore z = g(f(x)) = (g \circ f)(x)$ for $x \in A$. $\Rightarrow g \circ f$ is onto.

(b). Given $y \in f(A)$, there is $x \in A$ such that $y = f(x)$. If there is another $\tilde{x} \in A$ such that $y = f(\tilde{x})$, then

$$y = f(x) = f(\tilde{x}) \text{ and } f \text{ is } 1-1 \Rightarrow x = \tilde{x}$$

Therefore we can define $f^{-1} : f(A) \to A$ by the way that f^{-1} maps $y \in f(A)$ to the unique $x \in A$ which has y as its image. By the definition of f^{-1}, if $f^{-1}(y_1) = f^{-1}(y_2) = x \in A$, then $y_1 = f(x) = y_2$. $\therefore f^{-1}$ is 1-1. Now to prove that f^{-1} is onto. For $x \in A$, $f(x) \in f(A)$. By our definition of f^{-1}, we have $f^{-1}(f(x)) = x$ for $f(x) \in f(A)$. $\therefore f^{-1}$ is onto.

(c). **f is 1-1 $\Leftrightarrow$ g$\circ$f is 1-1** :

($\Rightarrow$) By (a).

($\Leftarrow$) If $f(x_1) = f(x_2)$ for some $x_1, x_2 \in A$. Then $(g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2)$ and $g \circ f$ is 1-1 $\Rightarrow x_1 = x_2$. $\therefore f$ is 1-1.

f is onto $\Leftrightarrow$ g$\circ$f is onto :

($\Rightarrow$) By (a).

($\Leftarrow$) For each $y \in B$, $g(y) \in C$ and $g \circ f$ is onto.

$$\Rightarrow \exists x \in A \quad \ni g(y) = (g \circ f)(x) = g(f(x))$$

. Then since g is 1-1 , $y = f(x)$. $\therefore f$ is onto.

11. (a). $n \in \mathbb{N}$, $q \in \mathbb{Q}$

(i) $\underline{q \geq 0}$. Then $q = \frac{r}{p}$ for some $p \in \mathbb{N}$, $r \in \mathbb{N} \cup \{0\}$.

$$(n^q)^p = (n^{\frac{r}{p}})^p = n^r$$

$\therefore n^q$ is a root of the polynomial $x^p - n^r$.

(Note that $\underline{p \in \mathbb{N}}$ and $r \in \mathbb{N} \cup \{0\} \Rightarrow \underline{n^r \in \mathbb{N}}$)

$\therefore n^q$ is algebraic.

(ii) $\underline{q < 0}$. Then $q = \frac{r}{p}$ for some $p \in -\mathbb{N}$, $r \in \mathbb{N}\{0\}$.

$$(n^r)(n^q)^{-p} = (n^r)(n^{\frac{r}{p}})^{-p} = (n^r)(n^{-r}) = 1.$$

$\therefore$ n^q is a root of the polynomial $n^r x^{-p} - 1$
(Note that $\underline{-p \in \mathbb{N}}$ and $\underline{n^r \in \mathbb{N}}$)
$\therefore$ n^q is algebraic.

(b). For each $n \in \mathbb{N}$, the set $\mathbb{Z}_n[x]$ of all polynomials with integer coefficients of degree n is countable. (By considering the choice of coefficients, it is easy to see that there is a 1-1 and onto function from $\mathbb{Z}_n[x]$ to $\mathbb{N}^n$. Hence $P_n[x]$ is countable.) And each of the polynomial has at most n roots. Hence the set of all roots of polynomials in $\mathbb{Z}_n[x]$ is countable. The collection of algebraic numbers of degree n is a subset of the previous set , hence it is countable too.

(c). Let $A_n =$ the collection of all algebraic numbers of degree n,
$A =$ the collection of all algebraic numbers,
$T =$ the collection of all transcendental numbers.
$\therefore$ $A = \bigcup_{n=1}^{\infty} A_n$ is countable. (Note that a countable union of countable sets is still countable.) Then since $\mathbb{R}$ is uncountable and $\mathbb{R}= A \cup T$, T is uncountable (or $\mathbb{R}= A \cup T$ is countable, a contradiction.).

HW §2.1

4. (a). Given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $|\, b_n - 0| = b_n < \epsilon$ if $n \geq N$. $\therefore$ For $n \geq N$, n large

$$\begin{array}{rrcl} & |x_n - a| & \leq & b_n \\ \Rightarrow & -\epsilon < -b_n \leq x_n - a & \leq & b_n < \epsilon \\ \Rightarrow & |x_n - a| & \leq & \epsilon \end{array}$$

i.e. $\lim_{n\to\infty} x_n = a$.

(b). Let $\tilde{b}_n = cb_n$, then $\tilde{b}_n \geq 0$ and $\lim_{n\to\infty} \tilde{b}_n = C \lim_{n\to\infty} b_n = C \cdot 0 = 0$.
And $|\, x_n - a| \leq Cb_n = \tilde{b}_n$. Hence $\{\tilde{b}_n\}$ satisfies the same condition as $\{b_n\}$ in (a).
$\therefore$ x_n converges to a.

6. (a). $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n \triangleq l$.
Given $\epsilon > 0$, there are $N_1, N_2 \in \mathbb{N}$ such that

$$|\, x_n - l| < \frac{\epsilon}{2} \quad \text{if} \;\; n \geq N_1$$

$$|\, y_n - l| < \frac{\epsilon}{2} \quad \text{if} \;\; n \geq N_2$$

Let $N = \max\{N_1, N_2\} \in \mathbb{N}$, then $n \geq N \Rightarrow$

$$\begin{array}{rcl} |\, x_n - y_n| & \leq & |\, x_n - l| + |l - y_n| \\ & < & \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad (\because n \geq N \geq N_1 \text{and} n \geq N \geq N_2) \\ & = & \epsilon \end{array}$$

$\therefore$ $x_n - y_n \to 0$ as $n \to \infty$.

(b). For each $M \in \mathbb{R}$, by Archimedean Principle , there is $n_M \in \mathbb{N}$ such that $n_0 > M$. Therefore, $\{n\}$ is not bounded above. $\Rightarrow \{n\}$ is not bounded. $\Rightarrow \{n\}$ does not converge by Theorem 2.8 .

(c). Let $x_n = n$, $y_n = n + \frac{1}{n}$, then $|\, x_n - y_n| = \frac{1}{n} \to 0$ as $n \to \infty$. But $\{x_n\} = \{n\}$ does not converge by (b). (Note that $\{y_n\}$ does not converge too.)

HW §2.2

5. Let $r_n = \frac{[10^n x]}{10^n} \in \mathbb{Q}$. Then $\{r_n\} \to x$ as $n \to \infty$.

10. (a). $0 \le y < \frac{1}{10^n} \Rightarrow 0 \le 10^{n+1}y < 10$. Let $A = \{k \in \mathbb{N} : \, k \le 10^{n+1}y\}$. Then $0 \in A \neq \phi$ and $\forall k \in A$, $k \le 10^{n+1}y < 10$ i.e. A has an upper bound 10. $\Rightarrow A$ has a supremum w. It is easy to see that $w \in A$. $\therefore 0 \le w \le 10^{n+1}y < 10$. ($\Rightarrow 0 \le w \le 9$.) w is the supremum of A, so $w+1 \notin A$. $\therefore w+1 > 10^{n+1}y$. Hence $w \le 10^{n+1}y < w+1$. This implies that

$$\frac{w}{10^{n+1}} \le y \le \frac{w}{10^{n+1}} + \frac{1}{10^{n+1}}$$

(b). $0 \le x < \frac{1}{10^0} = 1$. By (a), there is $0 \le x_1 \le 9$, $x_1 \in \mathbb{Z}$ such that

$$\frac{x_1}{10} \le x < \frac{x_1}{10} + \frac{1}{10}$$

$$\Rightarrow 0 \le x - \frac{x_1}{10} < \frac{1}{10}$$

If there are $x_1, x_2, \ldots, x_n \in \{0, 1, \ldots, 9\}$ such that

$$\sum_{k=1}^{n} \frac{x_k}{10^k} \le x < \sum_{k=1}^{n} \frac{x_k}{10^k} + \frac{1}{10^n}$$

$$\Rightarrow 0 \le x - \sum_{k=1}^{n} \frac{x_k}{10^k} < \frac{1}{10^n}$$

By (a), there is $x_{n+1} \in \mathbb{Z}$, $0 \le x_{n+1} \le 9$ such that

$$\frac{x_{n+1}}{10^{n+1}} \le x - \sum_{k=1}^{n} \frac{x_k}{10^k} < \frac{x_{n+1}}{10^{n+1}} + \frac{1}{10^{n+1}}$$

$$\sum_{k=1}^{n+1} \frac{x_k}{10^k} \le x < \sum_{k=1}^{n+1} \frac{x_k}{10^k} + \frac{1}{10^{n+1}}$$

By induction on n. q.e.d.

(c). According to (b), $\sum_{k=1}^{n} \frac{x_k}{10^k}$ is a increasing sequence on n, bounded above by x. $\Rightarrow$ $\sum_{k=1}^{n} \frac{x_k}{10^k}$ converges as $n \to \infty$. Let $n \to \infty$ in the inequality in (b), we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{x_k}{10^k} \le x \le \lim_{n\to\infty} \sum_{k=1}^{n} \frac{x_k}{10^k} + \lim_{n\to\infty} \frac{1}{10^n} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{x_k}{10^k}$$

$$\text{i.e. } \lim_{n\to\infty} \sum_{k=1}^{n} \frac{x_k}{10^k} = x$$

HW §2.3

4. Note that $x_n \leq 1$ for all $n \in \mathbb{N}$. $\Rightarrow x_n = 1 - \sqrt{1 - x_{n-1}} \geq 1 - \sqrt{1-1} = 0 \ \forall n \in \mathbb{N}$, $n \geq 2$.

$$\therefore 0 \leq x_n \leq 1 \quad \forall n \in \mathbb{N}$$

$$\begin{aligned} \sqrt{1-x_n} &= 1 - x_{n+1} \\ \Rightarrow \quad 1 - x_n &= 1 - 2x_{n+1} + x_{n+1}^2 \\ \Rightarrow \quad x_n - x_{n+1} &= x_{n+1}(1 - x_{n+1}) \geq 0 \end{aligned}$$

$$\therefore x_n \geq x_{n+1}$$

$\{x_n\}$ decreases and is bounded below by 0. Hence it converges to some $x \in \mathbb{R}$.
Let $n \to \infty$ in the equality $x_{n+1} = 1 - \sqrt{1 - x_n}$, we get

$$x = 1 - \sqrt{1-x}$$

$$\Rightarrow x = 0 \text{ or } 1$$

But x_n is decreasing, therefore $x \leq x_1 < 1$. $\therefore x = 0$. i.e. $x_n \downarrow 0$ as $n \to \infty$.

$$\begin{aligned} \frac{x_{n+1}}{x_n} &= \frac{1-\sqrt{1-x_n}}{x_n} \\ &= \frac{1-(1-x_n)}{x_n(1+\sqrt{1-x_n})} \\ &= \frac{1}{1+\sqrt{1-x_n}} \\ &\to \frac{1}{1+\sqrt{1-0}} = \frac{1}{2} \quad \text{as} \quad n \to \infty \end{aligned}$$

9. (a). For $n = 1$, $x_1 > y_1 > 0$. Suppose that $x_n > y_n > 0$ for some $n \in \mathbb{N}$, then

$$x_{n+1} = \frac{x_n + y_n}{2} > 0$$

$$y_{n+1} = \sqrt{x_n y_n} > 0$$

$$x_{n+1}^2 - y_{n+1}^2 = \frac{(x_n - y_n)^2}{4} > 0$$

$$\Rightarrow x_{n+1}^2 > y_{n+1}^2$$

$$\Rightarrow x_{n+1} > y_{n+1} > 0$$

By induction on n, $0 < y_n < x_n$ for all $n \in \mathbb{N}$.

(b).

$$x_{n+1} = \frac{x_n + y_n}{2} < \frac{x_n + x_n}{2} = x_n$$

$$y_{n+1} = \sqrt{x_n y_n} > \sqrt{y_n y_n} = y_n$$

$$y_1 < y_n < x_n < x_1 \quad \text{by (a)}$$

Therefore $\{x_n\}$ is strictly decreasing and bounded below by y_1 and $\{y_n\}$ is strictly increasing and bounded above by x_1. By Monotone Convergence Theorem , the two sequences converge.

(c).

$$0 < x_2 - y_2 = \frac{x_1 + y_1}{2} - \sqrt{x_1 y_1} < \frac{x_1 + y_1}{2} - \sqrt{y_1 y_1} = \frac{x_1 - y_1}{2}.$$

Suppose that $x_{n+1} - y_{n+1} < \frac{x_1 - y_1}{2^n}$, then

$$x_{n+2} - y_{x+2} = \frac{x_{n+1}}{y_{n+1}} - \sqrt{x_{n+1} y_{n+1}} < \frac{x_{n+1}}{y_{n+1}} - \sqrt{y_{n+1} y_{n+1}} = \frac{x_{n+1} - y_{n+1}}{2} < \frac{x_1 - y_1}{2^{n+1}}$$

By induction on n , $x_{n+1} - y_{n+1} < \frac{x_1 - y_1}{2^n}$ for all $n \in \mathbb{N}$.

(d). $\lim_{n\to\infty} \frac{x_1 - y_1}{2^{n-1}} = 0$, and $0 < x_n - y_n < \frac{x_1 - y_1}{2^{n-1}}$ for all $n \in \mathbb{N}$, $n \geq 2$.
$\Rightarrow 0 = \lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n = 0$. $\therefore \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

11. (a). Claim : $x_n > y_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$.

$$x_0 = 2\sqrt{3} > 3 = y_0 > 0$$

Suppose $x_n > y_n > 0$ for some $n \in \mathbb{N}$

$$x_{n+1} = \frac{2x_n y_n}{x_n + y_n} > \frac{2x_n y_n}{x_n + x_n} = y_n > 0$$

$$\therefore y_{n+1} = \sqrt{x_{n+1} y_n} < \sqrt{x_{n+1} x_{n+1}} = x_{n+1}$$

By induction on n, the Claim holds for all $n \in \mathbb{N} \cup \{0\}$
According to the Claim, for each $n \in \mathbb{N}$

$$x_n = \frac{2x_{n-1} y_{n-1}}{x_{n-1} + y_{n-1}} < \frac{2x_{n-1} y_{n-1}}{y_{n-1} + y_{n-1}} = x_{n-1}$$

$$y_n = \sqrt{x_n y_{n-1}} > \sqrt{y_n y_{n-1}} \Rightarrow \sqrt{y_n} > \sqrt{y_{n-1}} \Rightarrow y_n > y_{n-1}$$

$$\Rightarrow x_0 > x_n > y_n > y_0$$

$\therefore \{x_n\}$ decreases and is bounded below by y_0, hence has limit $x \in \mathbb{R}$; $\{y_n\}$ increases and is bounded above by x_0, hence has limit $y \in \mathbb{R}$

(b).

$$x_0 > x_n > y_n > y_0$$

$$2\sqrt{3} = x_0 \geq x \geq y \geq y_0 = 3 > 0$$

Let $n \to \infty$ in $x_n = \frac{2x_{n-1} y_{n-1}}{x_{n-1} + y_{n-1}}$ to get $x = \frac{2xy}{x+y}$.
$\Rightarrow x^2 + xy = 2xy \Rightarrow x^2 = xy \Rightarrow x = y$ ($\because x > 0$)