HW §1.2

7. Let $c = b - a > 0$, then

$$\begin{array}{rcll} b^n = (a+c)^n & \geq & a^n + na^{n-1}c & (\text{ by §1.2 Ex2}(b).\ ) \\ & > & a^n \geq 0 & (\because na^{n-1}c > 0\ ) \end{array}$$

Notice that $\sqrt[n]{a} \geq 0$ and $\sqrt[n]{b} > 0$ as the statements in page 6. Suppose that $\sqrt[n]{a} \geq \sqrt[n]{b} > 0$, then by the previous result,

$$a = (\sqrt[n]{a})^n \geq (\sqrt[n]{b})^n = b > 0$$

A contradiction to the fact that $a < b$. Hence $0 \leq \sqrt[n]{a} < \sqrt[n]{b}$ for all $n \in \mathbb{N}$.

8. (a).

$$\begin{array}{rrcll} & \sqrt{n+3} + \sqrt{n} & = & q \in \mathbb{Q} & (\text{ Obviously,} q > 0\ ) \\ \Rightarrow & \sqrt{n+3} & = & q - \sqrt{n} & \\ \Rightarrow & n+3 & = & q - 2q\sqrt{n} + n & \\ \Leftrightarrow & \sqrt{n} & = & \frac{q-3}{2q} \in \mathbb{Q} & (\because q \neq 0\ ) \\ \Rightarrow & n & = & k^2 \text{ for some } k \in \mathbb{N} & \\ \Rightarrow & \sqrt{n+3} & = & q - n \in \mathbb{Q} & \\ \Rightarrow & n+3 & = & l^2 \text{ for some } l \in \mathbb{N} & \end{array}$$

Solve

$$\left\{ \begin{array}{rcl} n & = & k^2 \\ n+3 & = & l^2 \end{array} \right. \quad n, k, l \in \mathbb{N}$$

to get

$$\left\{ \begin{array}{l} n = k = 1 \\ l = 2 \end{array} \right.$$

Substitute $n = 1$ into $\sqrt{n+3} + \sqrt{n}$ to verify that it is really a rational.

(b). Similar to (a) . (Solution: $n = 9$)

10. (a). To prove that $c_k - b_k = 1\ \forall\ k \in \mathbb{N}$ by induction on k.

$$b_1 = 2a_0 + b_0 + 2 = 12$$
$$c_1 = 2a_0 + c_0 + 2 = 13$$

$\therefore c_1 - b_1 = 1$. Suppose that $c_k - b_k = 1$ for some $k \in \mathbb{N}$, then

$$\begin{array}{rcl} c_{k+1} - b_{k+1} & = & (2a_k + c_k + 2) - (2a_k + b_k + 2) \\ & = & c_k - b_k \\ & = & 1. \end{array}$$

Hence $c_k - b_k = 1 \quad \forall\ k \in \mathbb{N}$.

(b). To prove the equality by induction on k. For $k = 1$, $a_1 = 5,\ b_1 = 12,\ c_1 = 13 \Rightarrow$ the equality $c_1^2 = a_1^2 + b_1^2$ is true. Suppose that $c_k^2 = a_k^2 + b_k^2$ for some $k \in \mathbb{N}$, then

$$\begin{aligned} c_{k+1}^2 - b_{k+1}^2 &= (c_{k+1}^2 - b_{k+1}^2)(c_{k+1}^2 + b_{k+1}^2) \\ &= 1 \cdot (c_{k+1}^2 + b_{k+1}^2) \\ &= 2a_k + c_k + 2 + 2a_k + b_k + 2 \\ &= 4 + 4a_k + c_k + b_k \\ &= 4 + 4a_k + (c_k + b_k)(c_k - b_k) \\ &= 4 + 4a_k + c_k^2 - b_k^2 \\ &= 4 + 4a_k + a_k^2 \\ &= (2 + a_k)^2 \\ &= a_{k+1}^2 \end{aligned}$$

i.e. $c_{k+1}^2 = a_{k+1}^2 + b_{k+1}^2$. Therefore, $c_k^2 = a_k^2 + b_k^2 \ \ \forall\ k \in \mathbb{N}$

§1.3

1. (f). $E = \{x \in \mathbb{R} \ :\ x = \frac{1}{n} - (-1)^n \text{ for } n \in \mathbb{N}\}$. Then $\sup E = \max E = 2$, $\inf E = -1$.

   Sol: For each $x \in \mathbb{R}$, $x = \frac{1}{n} - (-1)^n$ for some $n \in \mathbb{N}$.

   $$-1 = 0 + (-1) \le x = \frac{1}{n} - (-1)^n = \frac{1}{n} + (-1)^{n+1} \le 1 + 1 = 2$$

   $\therefore$ 2 is an upper bound of E , -1 is a lower bound of E .
   Since $2 = \frac{1}{1} - (-1)^1 \in E$, $2 = \sup E$ by §1.3 Ex7(a).
   For any $a > -1$,

   $$a - (-1) > 0 \Rightarrow \exists n_0 \in \mathbb{N} \ni .\ \tfrac{1}{n_0} < a - (-1) = a + 1$$

   $$\Rightarrow \frac{1}{2n_0} - (-1)^{2n_0} = \frac{1}{2n_0} - 1 < \frac{1}{n_0} - 1 < a + 1 - 1 = a$$

   And $\frac{1}{2n_0} - (-1)^{2n_0} \in E$. $\therefore$ a is not a lower bound of E if $a > -1$. $\Rightarrow a \le -1$ for any lower bound a of E . $\therefore \inf E = -1$.

   (g). $E = \{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$. Then $\sup E = \frac{3}{2}$, $\inf E = 0$.

   Sol: For each $n \in \mathbb{N}$ and $n \ge 2$, $\ \ (0 < \frac{1}{n} \le \frac{1}{2})$

   $$0 < \frac{1}{2} = 1 - \frac{1}{2} < 1 - \frac{1}{n} \le 1 + \frac{(-1)^n}{n} \le 1 + \frac{1}{n} \le 1 + \frac{1}{2} = \frac{3}{2}$$

   For $n = 1$,

   $$0 = 1 + \frac{(-1)^1}{1} < \frac{3}{2}$$

   $\therefore 0 \le 1 + \frac{(-1)^n}{n} \le \frac{3}{2}$ for all $n \in \mathbb{N}$.
   i.e. 0 is a lower bound of E , and $\frac{3}{2}$ is an upper bound of E . Note that $0 = 1 + \frac{(-1)^1}{\in} E$, $\frac{3}{2} = 1 + \frac{(-1)^2}{2} \in E$. $\therefore 0 = \inf E, \frac{3}{2} = \sup E$ by §1.3 Ex7.

3. $a < b$, $a, b \in \mathbb{R} \Rightarrow a - \sqrt{2} < b - \sqrt{2}$. By Theorem 1.24 (Density of rationals), there is a $q \in \mathbb{Q}$ such that $a - \sqrt{2} < q < b - \sqrt{2}$ ($\Rightarrow a < q + \sqrt{2} < b$) . Take $\xi = q + \sqrt{2}$, then $a < \xi < b$. Note that ξ is an irrational because it is a sum of a rational and an irrational (See §1.1 Ex6(c)).

5. (a). (Omitted)

   (b). By Theorem 1.28 (ii), E has a finite infimum $\Leftrightarrow$ $-E$ has a finite supremum, and

$$\sup(-E) = -\inf E$$

   $\therefore$ Given $\epsilon > 0$, there is $b \in (-E)$ such that

$$\begin{array}{rrcccl} & \sup(-E) - \epsilon & < & b & \le & \sup(-E) \\ \Leftrightarrow & -\inf E - \epsilon & < & b & \le & -\inf E \\ \Leftrightarrow & \inf E + \epsilon & > & -b & \ge & \inf E \end{array}$$

   Take $a = -b \in E$. q.e.d.

7. (a). $x \in E$, and x is an upper bound of $E \subset \mathbb{R}$. If M is an upper bound of E, then $M \ge x$ ($\because x \in E$). This implies $x = \sup E$.

   (b). If x is a lower bound of a set $E \subset \mathbb{R}$, and $x \in E$, the $x = \inf E$.
   The proof is similar to (a).

   (c). Let $E = (0, 1)$, then $\sup E = 1$, $\inf E = 0$, but $1 \notin E$, $0 \notin E$.

8. For each $n \in \mathbb{N}$,

$$|\ x_n| \le M \qquad \forall n \in \mathbb{N}$$

$\therefore \{x_n, x_{n+1}, \ldots\}$ is bounded above and nonempty.
$\Rightarrow s_n = \sup\{x_n, x_{n+1}, \ldots\}$ exists.

$$\begin{array}{rrcl} & \{x_n, x_{n+1}, \ldots\} & \supset & \{x_{n+1}, x_{n+2}, \ldots\} \\ \Rightarrow & s_n = \sup\{x_n, x_{n+1}, \ldots\} & \ge & \sup\{x_{n+1}, x_{n+2}, \ldots\} = s_{n+1} \end{array}$$

$\therefore \{s_n\}$ is decreasing.
The result about $\{t_n\}$ is similar to $\{s_n\}$