# HW §1.1

6. (a) <u>Lemma</u> : $(mn)^{-1} = m^{-1}n^{-1} = n^{-1}m^{-1}$

<u>pf</u> : $(mn)(n^{-1}m^{-1}) = m(nn^{-1})m^{-1} = m \cdot 1 \cdot m^{-1} = mm^{-1} = 1$

By the uniqueness of multiplicative inverse, we get $(mn)^{-1} = n^{-1}m^{-1} = m^{-1}n^{-1}$

- $\frac{m}{n} + \frac{p}{q} = mn^{-1} + pq^{-1} = m \cdot 1 \cdot n^{-1} + p \cdot 1 \cdot q^{-1} = m(qq^{-1})n^{-1} + p(nn^{-1})q^{-1}$ $= mq(q^{-1}n^{-1}) + pn(n^{-1}q^{-1}) = mq(nq)^{-1} + pn(nq)^{-1} = (mq+np)(nq)^{-1} = \frac{mq+np}{nq}$
- $\frac{m}{n} \cdot \frac{p}{q} = mn^{-1} \cdot pq^{-1} = m(n^{-1}p)q^{-1} = m(pn^{-1})q^{-1} = (mp)(n^{-1}q^{-1}) = (mp)(nq)^{-1}$ $= \frac{mp}{nq}$
- $\frac{m}{n} + \frac{-m}{n} = mn^{-1} + (-m)n^{-1} = (m + (-m))n^{-1} = 0 \cdot n^{-1} = 0$. By the uniqueness of additive inverse, we get $-\frac{m}{n} = \frac{-m}{n}$.
- $\frac{l}{n} \cdot \frac{n}{l} = ln^{-1} \cdot nl^{-1} = ll^{-1}nn^{-1} = 1 \cdot 1 = 1$. By the uniqueness of multiplicative inverse, we get $(\frac{l}{n})^{-1} = \frac{n}{l}$

(b) Since $\mathbb{Q} \subset \mathbb{R}$, the associative properties, commutative properties, distributive law are satisfied by the elements in $\mathbb{Q}$. The closure properties are true for $\mathbb{Q}$ from the first two equalities in (a). Now we have to show the existence and uniqueness of the additive identity and multiplicative identity. The existence is true just from the fact that the identities for $\mathbb{R}$ belong to $\mathbb{Q}$. Suppose that the identities are not unique, i.e. there exist another identities $0' \in \mathbb{Q}$ and $1' \in \mathbb{Q}$ for $\mathbb{Q}$. Then

$$0 = 0 + 0' = 0' \text{ and } 1 = 1 \cdot 1' = 1' .$$

Hence the identities are unique. Since we have shown that the identities for $\mathbb{Q}$ are just the ones for $\mathbb{R}$, the existence of the inverses comes from the last two equalities in (a). (The equalities show that the inverses are in $\mathbb{Q}$) And their uniqueness are just because the uniqueness of the inverses in $\mathbb{R}$.

(c) Suppose that there are $p, q \in \mathbb{Q}$ , $r \in \mathbb{R}\backslash\mathbb{Q}$ such that $p + r = q$. Then $r = 0 + r = (-p) + p + r = (-p) + q \in \mathbb{Q}$. ( Note that from (b), $p \in \mathbb{Q} \Rightarrow (-\mathbb{P}) \in \mathbb{Q}$ and the sum of rational numbers is still rational. ) This contradicts to the fact that $r \in \mathbb{R}\backslash\mathbb{Q}$. Hence the sum of a rational and an irrational is always irrational.
But the product of a rational and an irrational may be rational or irrational. We show that by the examples : $0 \cdot \sqrt{2} = 0 \in \mathbb{Q}$ , $1 \cdot \sqrt{2} = \sqrt{2} \in \mathbb{R}\backslash\mathbb{Q}$

(d) <u>Lemma</u> : $n > 0 \Rightarrow n^{-1} > 0$

<u>pf</u> : Suppose the contrary, i.e. $n^{-1} \leq 0$. Note first that $n^{-1}$ can't be zero , because $1 = nn^{-1} \neq n^0 = 0$. Hence $n^{-1} < 0$. Then

$$n > 0, n^{-1} < 0 \Rightarrow 1 = nn^{-1} < n \cdot 0 = 0.$$

This contradicts to that $1 > 0$. $\therefore n^{-1} > 0$.

$n, q > 0 \Rightarrow nq, n^{-1}, q^{-1}, n^{-1}q^{-1} > 0$.
Therefore,

$$\begin{aligned} & \tfrac{m}{n} < \tfrac{p}{q} \\ \Leftrightarrow \quad & mn^{-1} < pq^{-1} \\ \Rightarrow \quad & mn^{-1}nq < pq^{-1}nq \\ \Leftrightarrow \quad & mq < np \end{aligned}$$

$$\begin{aligned} & mq < np \\ \Rightarrow\quad & mqn^{-1}q^{-1} < npn^{-1}q^{-1} \\ \Leftrightarrow\quad & mn^{-1} < pq^{-1} \\ \Leftrightarrow\quad & \tfrac{m}{n} < \tfrac{p}{q} \end{aligned}$$

10. (a)

$$\begin{aligned} |xy - ab| &= \|(x-a)(y-b) + a(y-b) + b(x-a)| \\ &\leq |(x-a)(y-b)| + |a(y-b)| + |b(x-a)| \\ &= |x-a||y-b| + |a||y-b| + |b||x-a| \\ &< \varepsilon \cdot \varepsilon + |a|\varepsilon + |b|\varepsilon \\ &= (|a| + |b|)\varepsilon + \varepsilon^2 \end{aligned}$$

(b)

$$\begin{aligned} |x^2y - a^2b| &= |(x-a)^2(y-b) + b(x-a)^2 + 2a(x-a)(y-b) + 2ab(x-a) + a^2(y-b)| \\ &\leq |x-a|^2|y-b| + |b||x-a|^2 + 2|a||x-a||y-b| + 2|ab||x-a| + |a|^2|y-b| \\ &< \varepsilon^2 \cdot \varepsilon + |b|\varepsilon^2 + 2|a|\varepsilon \cdot \varepsilon + 2|ab|\varepsilon + |a|^2\varepsilon \\ &= \varepsilon(|a|^2 + 2|ab|) + \varepsilon^2(|b| + 2|a|) + \varepsilon^3 \end{aligned}$$

11. (a) (i) By Trichotomy property, for $x, 0 \in \mathbb{R}$, one and only one of the following statements holds:

$$0 < x, \quad x < 0 (\Leftrightarrow -x > 0), \quad \text{or} \quad 0 = x$$

i.e. one and only one of the following statements holds:

$$x \in \mathbb{R}^+ \;, \quad -x \in \mathbb{R}^+ \;, \quad \text{or} \quad x = 0$$

(ii)

$$x > 0, y > 0 \Rightarrow x + y > x + 0 > 0 + 0 = 0 \quad \text{and} \quad x \cdot y > x \cdot 0 = 0$$

i.e. both $x + y$ and $x \cdot y$ belong to $\mathbb{R}$.

(b) **Trichotomy Property.** Given $a, b \in \mathbb{R}$, by (a)(i) one and only one of the statements holds: $a-b \in \mathbb{R}^+$, $-(a-b) = b-a \in \mathbb{R}^+$, or $a-b = 0$. The statements are equivalent to $a \prec b$, $b \prec a$, $a = b$ respectively. Hence trichotomy property holds.
**Transitive Property.** Suppose that $a \prec b$ and $b \prec c$, i.e. $a-b \in \mathbb{R}^+$ and $b-c \in \mathbb{R}^+$. (a)(ii)$\Rightarrow$ $(a-b) + (b-c) = a-c \in \mathbb{R}^+$, i.e. $a \prec c$.
**Additive Property.** Suppose that $a \prec b$ and $c \in \mathbb{R}$. Then $(a+c) - (b+c) = a - b \in \mathbb{R}^+$. $\therefore a + c \prec b + c$.
**Multiplicative Properties.** Suppose that $a \prec b$ and $c \succ 0$. Then $a - b \in \mathbb{R}^+$ and $c \in \mathbb{R}^+$. (a)(ii) $\Rightarrow ac - bc = (a-b) \cdot c \in \mathbb{R}^+$, i.e. $ac \prec bc$. Similarly, if $a \prec b$ and $c \prec 0$. Then $a-b \in \mathbb{R}^+$ and $-c = 0-c \in \mathbb{R}^+$.(a)(ii) $\Rightarrow bc - ac = (a-b) \cdot (-c) \in \mathbb{R}^+$, i.e. $bc \prec ac$.
As shown above, Postulate 2 holds with $\prec$ in place of $<$.