§8.3 # 4

(a) A union of open sets is open. So if \( V = \bigcup_{\alpha \in A} B_\alpha \) with \( B_\alpha \) open, then \( V \) is open. For the other direction, we assume that \( V \) is open, then for \( x \in V \), there exists \( B_{r_x}(x) \subset V \) for some \( r_x > 0 \). Since \( B_{r_x}(x) \) is a subset of \( V \) for all \( x \in V \), we have \( \bigcup_{x \in V} B_{r_x}(x) \subset V \). On the other hand, for any \( x \in V \), we must have \( x \in B_{r_x}(x) \). Thus, \( V \in \bigcup_{x \in V} B_{r_x}(x) \).

(b) Using the complement argument, we can see that if \( V \) is closed then there is a collection of \( B_\alpha^c \) for \( \alpha \in A \) such that \( V = \bigcap_{\alpha \in A} B_\alpha^c \).

§8.3 # 5

\( E \) is closed iff \( E^c \) is open. If \( a \notin E \) then \( x \in E^c \). So there exists \( \varepsilon > 0 \) such that \( B_\varepsilon(x) \subset E^c \). Therefore, \( \|x - a\| \geq \varepsilon > 0 \) for all \( x \in E \). Hence inf_{x \in E} \|x - a\| > 0.

§8.3 # 8

(a) If \( C \) is closed in \( \mathbb{R}^n \) and \( C = C \cap E \) (\( C \) is a subset of \( E \)), then \( C \) is relatively closed in \( E \). On the other hand, if \( C \) is relatively open in \( E \), then there exists a closed set \( B \) such that \( C = B \cap E \). Hence \( C^c = B^c \cup E^c \). For both \( B^c \) and \( E^c \) are open, we know that \( C^c \) is open. So \( C \) is closed.

(b) \( C \) is relatively closed iff there exists a closed set \( B \) such that \( C = B \cap E \). Thus, we have \( E \setminus C = E \cap B^c \). So \( E \setminus C \) is relatively open. The proof of the other direction is similar.

§8.3 # 9

Assume that \( E \) is not connected. That is, there exist open sets \( U \) and \( V \) such that \( U \cap E \neq \emptyset \), \( V \cap E \neq \emptyset \), \( U \cap V = \emptyset \), and \( E \subseteq U \cup V \). From \( E = \bigcup_{\alpha \in A} E_\alpha \) we have \( U \cap (\bigcup_{\alpha \in A} E_\alpha) = \bigcup_{\alpha \in A} (U \cap E_\alpha) \neq \emptyset \). Therefore \( U \cap E_{\alpha_1} \neq \emptyset \) for some \( \alpha_1 \in A \). Likewise, we can show that \( V \cap E_{\alpha_2} \neq \emptyset \) for some \( \alpha_2 \in A \). If \( \alpha_1 = \alpha_2 = \alpha \), then \( E_\alpha \) is not connected. Then we have a contradiction. The problem is that \( \alpha_1 \) is not necessarily equal to \( \alpha_2 \). However, in fact, we want to show that either \( U \cap E_\alpha \neq \emptyset \) or \( V \cap E_\alpha \neq \emptyset \) for all \( \alpha \in A \). Assume not, namely, \( U \cap E_\alpha = \emptyset \) for some \( \alpha \in A \) and \( V \cap E_{\alpha'} = \emptyset \) for some \( \alpha' \in A \). Thus, \( U \cap \bigcap_{\alpha \in A} E_\alpha = \emptyset \) and \( V \cap \bigcap_{\alpha \in A} E_\alpha = \emptyset \). This implies that \( E \subseteq U \cup V \) is not true since \( \bigcap_{\alpha \in A} E_\alpha \neq \emptyset \). So we must have \( U \cap E_\alpha \neq \emptyset \) and \( V \cap E_\alpha \neq \emptyset \) for some \( \alpha \in A \). Additionally, \( E_\alpha \subseteq E \subseteq U \cup V \). Therefore, \( E_\alpha \) is not connected. This is a contradiction.

§8.3 # 10

(i) Any connected set in \( \mathbb{R} \) is an interval (single point is included). If \( \emptyset \neq E = \bigcap_{\alpha \in A} E_\alpha \) is not connected, then there exists at least one point \( c \) in \( E \) such that \( (a, c) \subset E \) and \( (c, b) \subset E \). In other words, \( (a, c) \subset E_\alpha \) and \( (c, b) \subset E_\alpha \) for all \( \alpha \). Hence \( E_\alpha \) is not connected. This is a contradiction.
(ii) Let $E_1 = \{(x, y) : x^2 + y^2 = 1, y \geq 0\}$ and $E_2 = \{(x, y) : y = 0\}$. Then $E_1 \cap E_2 = \{(-1, 0), (1, 0)\}$. Both $E_1$ and $E_2$ are connected, but $E_1 \cap E_2$ is not connected.

§8.4 # 2

(a) $E^0 = \{(x, y) : x^2 + 4y^2 < 1\}$, $\overline{E} = E$, and $\partial E = \{(x, y) : x^2 + 4y^2 = 1\}$.

(b) $E^0 = \emptyset$, $\overline{E} = E$, and $\partial E = E$.

(c) $E^0 = \{(x, y) : y > x^2, y < 1\}$, $\overline{E} = \{(x, y) : y \geq x^2, 0 \leq y \leq 1\}$, $\partial E = \{(x, y) : y = x^2, -1 \leq x \leq 1\} \cup \{(x, y) : -1 < x < 1, y = 1\}$.

(d) $E^0 = E$, $\overline{E} = \{(x, y) : x^2 - y^2 \leq 1, -1 \leq y \leq 1\}$, and $\partial E = \{(x, y) : x^2 - y^2 = 1, -1 \leq y \leq 1\} \cup \{(x, y) : -\sqrt{2} < x < \sqrt{2}, y = 1\} \cup \{(x, y) : -\sqrt{2} < x < \sqrt{2}, y = -1\}$

§8.4 # 3

$A \subseteq B \subseteq \overline{B} \Rightarrow \overline{A} \subseteq \overline{B}$; $A^0 \subseteq A \subseteq B \Rightarrow A^0 \subseteq B^0$.

§8.4 # 7

Suppose that $A$ is not connected. Then there exist two open sets $U$ and $V$ such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, $U \cap V = \emptyset$, and $A \subseteq U \cup V$. It is clear that $E \subseteq U \cup V$. Since $A \subseteq \overline{E}$, we have $U \cap \overline{E} \neq \emptyset$ and $V \cap \overline{E} \neq \emptyset$. Then both $U \cap E$ and $V \cap E$ are non-empty. For if $U \cap E = \emptyset$ then $E \subseteq U^c$. Since $U^c$ is closed, we get $\overline{E} \subseteq U^c$, i.e., $U \cap \overline{E} = \emptyset$. This is again a contradiction. Similar proof works for $V \cap E$. Now because $U \cap E \neq \emptyset$ and $V \cap E \neq \emptyset$, $E$ is not connected. This is a contradiction. So $A$ must be connected.

§8.4 # 8 Note that we use the canonical metric in $\mathbb{R}^n$ here.

(a) $\emptyset$ and $\mathbb{R}^n$.

(b) We assume that $E$ is connected. We know that $\emptyset$ and $E$ are relatively clopen sets. Assume that $E$ contains another relatively clopen set, say $U$. Then $E \setminus U$ is relatively open. So $E = U \cup U^c$ and $U \cap U^c = \emptyset$. Thus $E$ is not connected. This is a contradiction.

On the other hand, if $E$ has only two relatively clopen sets, i.e. $\emptyset$ and $E$, and $E$ is not connected. Hence, there exist relatively open sets $U$ and $V$ such that $E = U \cup V$ and $U \cap V = \emptyset$. So $V$ and $U$ are relatively clopen. This is a contradiction.

(c) If $\partial E = \emptyset$, then $E^0 = \overline{E}$. Hence $E$ is clopen. Hence $E$ and $E^c$ are open and $\mathbb{R}^n = E \cup E^c$. Also, $E \neq \emptyset$, $E^c \neq \emptyset$, and $E \cap E^c = \emptyset$. So $\mathbb{R}^n$ is not connected. This is a contradiction since $\mathbb{R}^n$ is connected.

§8.4 # 10 Answers can be found in the proof of Theorem 10.40.

§8.4 # 11
(a) $U$ is relatively open iff $\exists$ open set $\Omega$ in $\mathbb{R}^n$ such that $U = E \cap \Omega$. Since $U \subset E^0$, we have $U = E^0 \cap \Omega$, i.e. $U$ is open in $\mathbb{R}^n$. Thus, $U \cap \partial U = \emptyset$.

(b) If $x \in U \cap \partial E$, then $x \in U$ and $B_r(x) \cap E \neq \emptyset$, $B_r(x) \cap E^c \neq \emptyset$ for all $r > 0$. From $U \subset E$, we know that $E^c \subset U^c$. So $B_r(x) \cap U^c \neq \emptyset$. That $x \in U$ implies $B_r(x) \cap U \neq \emptyset$ for all $r > 0$. Therefore, we have $U \cap \partial E = U \cap \partial U$. 

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