§8.3 # 4

(a) A union of open sets is open. So if $V = \bigcup_{\alpha \in A} B_\alpha$ with B_α open, then V is open. For the other direction, we assume that V is open, then for $x \in V$, there exists $B_{r_x}(x) \subset V$ for some $r_x > 0$. Since $B_{r_x}(x)$ is a subset of V for all $x \in V$, we have $\bigcup_{x \in V} B_{r_x}(x) \subseteq V$. On the other hand, for any $x \in V$, we must have $x \in B_{r_x}(x)$. Thus, $V \in \bigcup_{x \in V} B_{r_x}(x)$.

(b) Using the complement argument, we can see that if V is closed then there is a collection of B_α^c for $\alpha \in A$ such that $V = \bigcap_{\alpha \in A} B_\alpha^c$.

§8.3 # 5

E is closed iff E^c is open. If $a \notin E$ then $x \in E^c$. So there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset E^c$. Therefore, $\|x - a\| \geq \varepsilon > 0$ for all $x \in E$. Hence $\inf_{x \in E} \|x - a\| > 0$.

§8.3 # 8

(a) If C is closed in $\mathbb{R}^n$ and $C = C \cap E$ (C is a subset of E), then C is relatively closed in E. On the other hand, if C is relatively open in E, then there exists a closed set B such that $C = B \cap E$. Hence $C^c = B^c \cup E^c$. For both B^c and E^c are open, we know that C^c is open. So C is closed.

(b) C is relatively closed iff there exists a closed set B such that $C = B \cap E$. Thus, we have $E \setminus C = E \cap B^c$. So $E \setminus C$ is relatively open. The proof of the other direction is similar.

§8.3 # 9

Assume that E is not connected. That is, there exist open sets U and V such that $U \cap E \neq \emptyset$, $V \cap E \neq \emptyset$, $U \cap V = \emptyset$, and $E \subseteq U \cup V$. From $E = \bigcup_{\alpha \in A} E_\alpha$ we have $U \cap (\bigcup_{\alpha \in A} E_\alpha) = \bigcup_{\alpha \in A}(U \cap E_\alpha) \neq \emptyset$. Therefore $U \cap E_{\alpha_1} \neq \emptyset$ for some $\alpha_1 \in A$. Likewise, we can show that $V \cap E_{\alpha_2} \neq \emptyset$ for some $\alpha_2 \in A$. If $\alpha_1 = \alpha_2 = \alpha$, then E_α is not connected. Then we have a contradiction. The problem is that α_1 is not necessarily equal to α_2. However, in fact, we want to show that either $U \cap E_\alpha \neq \emptyset$ or $V \cap E_\alpha \neq \emptyset$ for all $\alpha \in A$. Assume not, namely, $U \cap E_\alpha = \emptyset$ for some $\alpha \in A$ and $V \cap E_{\alpha'} = \emptyset$ for some $\alpha' \in A$. Thus, $U \cap \bigcap_{\alpha \in A} E_\alpha = \emptyset$ and $V \cap \bigcap_{\alpha \in A} E_\alpha = \emptyset$. This implies that $E \subseteq U \cup V$ is not true since $\bigcap_{\alpha \in A} E_\alpha \neq \emptyset$. So we must have $U \cap E_\alpha \neq \emptyset$ and $V \cap E_\alpha \neq \emptyset$ for some $\alpha \in A$. Additionally, $E_\alpha \subseteq E \subseteq U \cup V$. Therefore, E_α is not connected. This is a contradiction.

§8.3 # 10

(i) Any connected set in $\mathbb{R}$ is an interval (single point is included). If $\emptyset \neq E = \bigcap_{\alpha \in A} E_\alpha$ is not connected, then there exists at least one point c in E such that $(a, c) \subset E$ and $(c, b) \subset E$. In other words, $(a, c) \subset E_\alpha$ and $(c, b) \subset E_\alpha$ for all α. Hence E_α is not connected. This is a contradiction.

(ii) Let $E_1 = \{(x,y) : x^2 + y^2 = 1, y \geq 0\}$ and $E_2 = \{(x,y) : y = 0\}$. Then $E_1 \cap E_2 = \{(-1,0),(1,0)\}$. Both E_1 and E_2 are connected, but $E_1 \cap E_2$ is not connected.

<u>§8.4 # 2</u>

(a) $E^0 = \{(x,y) : x^2 + 4y^2 < 1\}$, $\overline{E} = E$, and $\partial E = \{(x,y) : x^2 + 4y^2 = 1\}$.

(b) $E^0 = \emptyset$, $\overline{E} = E$, and $\partial E = E$.

(c) $E^0 = \{(x,y) : y > x^2, y < 1\}$, $\overline{E} = \{(x,y) : y \geq x^2, 0 \leq y \leq 1\}$, $\partial E = \{(x,y) : y = x^2, -1 \leq x \leq 1\} \cup \{(x,y) : -1 < x < 1, y = 1\}$.

(d) $E^0 = E$, $\overline{E} = \{(x,y) : x^2 - y^2 \leq 1, -1 \leq y \leq 1\}$, and $\partial E = \{(x,y) : x^2 - y^2 = 1, -1 \leq y \leq 1\} \cup \{(x,y) : -\sqrt{2} < x < \sqrt{2}, y = 1\} \cup \{(x,y) : -\sqrt{2} < x < \sqrt{2}, y = -1\}$

<u>§8.4 # 3</u>

$A \subseteq B \subseteq \overline{B} \Rightarrow \overline{A} \subseteq \overline{B}$; $A^0 \subseteq A \subseteq B \Rightarrow A^0 \subseteq B^0$.

<u>§8.4 # 7</u>

Suppose that A is not connected. Then there exist two open sets U and V such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, $U \cap V = \emptyset$, and $A \subseteq U \cup V$. It is clear that $E \subseteq U \cup V$. Since $A \subseteq \overline{E}$, we have $U \cap \overline{E} \neq \emptyset$ and $V \cap \overline{E} \neq \emptyset$. Then both $U \cap E$ and $V \cap E$ are non-empty. For if $U \cap E = \emptyset$ then $E \subset U^c$. Since U^c is closed, we get $\overline{E} \subseteq U^c$, i.e., $U \cap \overline{E} = \emptyset$. This is again a contradiction. Similar proof works for $V \cap E$. Now because $U \cap E \neq \emptyset$ and $V \cap E \neq \emptyset$, E is not connected. This is a contradiction. So A must be connected.

<u>§8.4 # 8</u> Note that we use the canonical metric in $\mathbb{R}^n$ here.

(a) $\emptyset$ and $\mathbb{R}^n$.

(b) We assume that E is connected. We know that $\emptyset$ and E are relatively clopen sets. Assume that E contains another relatively clopen set, say U. Then $E \setminus U$ is relatively open. So $E = U \cup U^c$ and $U \cap U^c = \emptyset$. Thus E is not connected. This is a contradiction.

On the other hand, if E has only two relatively clopen sets, i.e. $\emptyset$ and E, and E is not connected. Hence, there exist relatively open sets U and V such that $E = U \cup V$ and $U \cap V = \emptyset$. So V and U are relatively clopen. This is a contradiction.

(c) If $\partial E = \emptyset$, then $E^0 = \overline{E}$. Hence E is clopen. Hence E and E^c are open and $\mathbb{R}^n = E \cup E^c$. Also, $E \neq \emptyset$, $E^c \neq \emptyset$, and $E \cap E^c = \emptyset$. So $\mathbb{R}^n$ is not connected. This is a contradiction since $\mathbb{R}^n$ is connected.

<u>§8.4 # 10</u> Answers can be found in the proof of Theorem 10.40.

<u>§8.4 # 11</u>

(a) U is relatively open iff $\exists$ open set Ω in $\mathbb{R}^n$ such that $U = E \cap \Omega$. Since $U \subset E^0$, we have $U = E^0 \cap \Omega$, i.e. U is open in $\mathbb{R}^n$. Thus, $U \cap \partial U = \emptyset$.

(b) If $x \in U \cap \partial E$, then $x \in U$ and $B_r(x) \cap E \neq \emptyset$, $B_r(x) \cap E^c \neq \emptyset$ for all $r > 0$. From $U \subset E$, we know that $E^c \subset U^c$. So $B_r(x) \cap U^c \neq \emptyset$. That $x \in U$ implies $B_r(x) \cap U \neq \emptyset$ for all $r > 0$. Therefore, we have $U \cap \partial E = U \cap \partial U$.