§14.4

#4.(a) By Abel's formula,

$$\sum_{k=0}^{n} a_k r^k = S_n r^n - \sum_{k=0}^{n-1} S_k (r^{k+1} - r^k)$$
$$= S_n r^n + (1-r)\sum_{k=0}^{n-1} S_k r^k \qquad (*)$$

Claim: $S_n r^n \to 0$ as $n \to \infty$ if one of $\sum_{k=0}^{\infty} a_k r^k$ and $\sum_{k=0}^{\infty} S_k r^k$ converge for all $r \in (0,1)$.

pf: In case $\sum_{k=0}^{\infty} a_k r^k$ converges $\forall r \in (0,1)$, $\sum_{k=0}^{\infty} a_k r^k$ converges absolutely.

$\therefore \sum_{k=0}^{n} |a_k| r^k$ is bdd by some $M > 0$.

Given $\varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t.

$$0 \le \sum_{k=n+1}^{n+p} |a_k| r^k < \frac{\varepsilon}{2} \quad \text{if } n \ge N_1,\ p \in \mathbb{N}.$$

$r^n \to 0$ as $n \to \infty \Rightarrow \exists N_2 \in \mathbb{N}$ s.t.

$$0 \le r^n < \frac{\varepsilon}{2M} \quad \text{if } n \ge N_2.$$

Then if $n \ge N_1 + N_2 + 1$,

$$0 \le |S_n r^n| \le \sum_{k=0}^{n} |a_k| r^n$$
$$= \sum_{k=0}^{N_1} |a_k| r^k r^{n-k} + \sum_{k=N_1+1}^{n} |a_k| r^k r^{n-k}$$
$$\le \sum_{k=0}^{N_1} |a_k| r^k \cdot \frac{\varepsilon}{2M} + \sum_{k=N_1+1}^{n} |a_k| r^k$$

$(\because r^{n-k} \le 1$ & for $0 \le k \le N_1$, $n-k \ge N_2$ $\therefore r^{n-k} < \frac{\varepsilon}{2M}$ for $0 \le k \le N_1)$

$$< M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore S_n r^n \to 0$ as $n \to \infty$.

In case $\sum_{k=0}^{\infty} S_k r^k$ converges, $S_k r^k \to 0$ as $k \to \infty$ obviously.

By the Claim and $(*)$, $\sum_{k=0}^{\infty} a_k r^k$ converges $\forall r \in (0,1)$ iff $\sum_{k=0}^{\infty} S_k r^k$ converges $\forall r \in (0,1)$. And $\sum_{k=0}^{\infty} a_k r^k = (1-r)\sum_{k=0}^{\infty} S_k r^k$ in the case.

Since $(k+1)\sigma_k = S_0 + S_1 + \cdots + S_k$, we can just apply the previous result to get that $\sum_{k=0}^{\infty} S_k r^k$ converges $\forall r \in (0,1)$ iff $\sum_{k=0}^{\infty} (k+1)\sigma_k r^k$ converges $\forall r \in (0,1)$. And $\sum_{k=0}^{\infty} S_k r^k = (1-r)\sum_{k=0}^{\infty} (k+1)\sigma_k r^k$ in the case.

$\therefore \sum_{k=0}^{\infty} a_k r^k = (1-r)\sum_{k=0}^{\infty} S_k r^k = (1-r)^2 \sum_{k=0}^{\infty} (k+1)\sigma_k r^k$ if one of the series converges $\forall r \in (0,1)$.

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#4(b) $\sigma_k \longrightarrow L$ as $k \longrightarrow \infty$

$\therefore \sigma_k$ is bdd. by some $M>0$.

$\therefore \sum_{k=0}^{\infty} |(k+1)\sigma_k r^k| \le M \sum_{k=0}^{\infty}(k+1)r^k = \frac{M}{(1-r)^2} < \infty$ for each $r \in (0,1)$

$\therefore \sum_{k=0}^{\infty}(k+1)\sigma_k r^k$ converges for all $r\in(0,1)$

By (a), $\sum_{k=0}^{\infty} a_k r^k$ converges for all $r \in (0,1)$, and

$$\sum_{k=0}^{\infty} a_k r^k = (1-r)^2 \sum_{k=0}^{\infty}(k+1)\sigma_k r^k.$$

Claim: $\sum_{k=0}^{\infty} a_k r^k = (1-r)^2 \sum_{k=0}^{\infty}(k+1)\sigma_k r^k \longrightarrow L$ as $r \to 1-$

pf: $\sigma_k \longrightarrow L$ as $k \longrightarrow \infty$.

$\therefore$ For each $\varepsilon>0$, $\exists N \in \mathbb{N}$ s.t

$$L-\varepsilon < \sigma_k < L+\varepsilon \quad \text{if} \quad k \ge N.$$

$$\sum_{k=N}^{\infty}(k+1)(L-\varepsilon)r^k \le \sum_{k=N}^{\infty}(k+1)\sigma_k r^k \le \sum_{k=N}^{\infty}(k+1)(L+\varepsilon)r^k$$

$$\sum_{k=N}^{\infty}(k+1)(L-\varepsilon)r^k = (L-\varepsilon)\sum_{k=N}^{\infty}(k+1)r^k = (L-\varepsilon)\left(\frac{(N+1)r^N}{1-r} + \frac{r^{N+1}}{(1-r)^2}\right)$$

$$\sum_{k=N}^{\infty}(k+1)(L+\varepsilon)r^k = (L+\varepsilon)\sum_{k=N}^{\infty}(k+1)r^k = (L+\varepsilon)\left(\frac{(N+1)r^N}{1-r} + \frac{r^{N+1}}{(1-r)^2}\right)$$

$\therefore (L-\varepsilon)((N+1)r^N(1-r)+r^{N+1}) + (1-r)^2\sum_{k=0}^{N-1}(k+1)\sigma_k r^k$

$\le (1-r)^2 \sum_{k=0}^{\infty}(k+1)\sigma_k r^k$

$\le (L+\varepsilon)((N+1)r^N(1-r)+r^{N+1}) + (1-r)^2\sum_{k=0}^{N-1}(k+1)\sigma_k r^k$

Since $\lim_{r\to 1-}((N+1)r^N(1-r)+r^{N+1}) = 1$

& $\lim_{r\to 1-}(1-r)^2\sum_{k=0}^{N-1}(k+1)\sigma_k r^k = 0$

$\therefore L-\varepsilon \le \underline{\lim}_{r\to 1-}(1-r)^2\sum_{k=0}^{\infty}(k+1)\sigma_k r^k \le \overline{\lim}_{r\to 1-}(1-r)^2\sum_{k=0}^{\infty}(k+1)\sigma_k r^k \le L+\varepsilon$

ε arbitrary $\Longrightarrow L \le \underline{\lim}_{r\to 1-}(1-r)^2\sum_{k=0}^{\infty}(k+1)\sigma_k r^k \le \overline{\lim}_{r\to 1-}(1-r)^2\sum_{k=0}^{\infty}(k+1)\sigma_k r^k \le L$

$$\Longrightarrow \lim_{r\to 1-}(1-r)^2\sum_{k=0}^{\infty}(k+1)\sigma_k r^k = L$$

$$\lim_{r\to 1-}(1-r)^2\sum_{k=0}^{\infty}(k+1)\sigma_k r^k = \lim_{r\to 1-}\sum_{k=0}^{\infty} a_k r^k$$

i.e. $\sum a_k$ is Abel summable to L

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#4.(c). f is periodic of period 2π, conti. and of bounded variation

By Thm 14.30, $\lim_{N\to\infty} S_N = f$ uniformly on $\mathbb{R}$.

i.e. $\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k \sin kx) = f(x)$ uniformly on $\mathbb{R}$

$\Rightarrow$ $\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k \sin kx)r^k$ converges $\forall r\in(0,1)$.

Given $\varepsilon>0$, $\exists N\in\mathbb{N}$ s.t. if $n\geq N$, $x\in\mathbb{R}$, $r\in(0,1)$, then

$$\left|\sum_{k=N}^{n}(a_k\cos kx + b_k\sin kx)\right| < \frac{\varepsilon}{3}$$

$$\therefore \left|\sum_{k=N}^{\infty}(a_k\cos kx + b_k\sin kx)\right| \leq \frac{\varepsilon}{3} \quad \text{if } x\in\mathbb{R},\ r\in(0,1).$$

By Abel's formula, for $n\geq N$, $x\in\mathbb{R}$, $r\in(0,1)$,

$$\left|\sum_{k=N}^{n}(a_k\cos kx + b_k\sin kx)r^k\right|$$

$$= \left|r^n\sum_{k=N}^{n}(a_k\cos kx + b_k\sin kx) - \sum_{k=N}^{n-1}\left(\sum_{j=N}^{k}(a_j\cos jx + b_j\sin jx)\right)(r^{k+1}-r^k)\right|$$

$$\leq r^n\left|\sum_{k=N}^{n}(a_k\cos kx + b_k\sin kx)\right| + \sum_{k=N}^{n-1}\left|\sum_{j=N}^{k}(a_j\cos jx + b_j\sin jx)\right||r^{k+1}-r^k|$$

$$< r^n\cdot\frac{\varepsilon}{3} + \sum_{k=N}^{n-1}\frac{\varepsilon}{3}(r^k-r^{k+1}) = \frac{\varepsilon}{3}r^N < \frac{\varepsilon}{3}$$

Let $n\to\infty$, we get that if $x\in\mathbb{R}$, $r\in(0,1)$,

$$\left|\sum_{k=N}^{\infty}(a_k\cos kx + b_k\sin kx)r^k\right| \leq \frac{\varepsilon}{3}$$

Since $\sum_{k=1}^{N-1}(|a_k|+|b_k|)r^k \longrightarrow \sum_{k=1}^{N-1}(|a_k|+|b_k|)$ as $r\to 1-$,

$\therefore \exists\ \delta>0$ s.t. $0 \leq \sum_{k=1}^{N-1}(|a_k|+|b_k|)(1-r^k) < \frac{\varepsilon}{3}$ if $\delta<r<1$.

$$\therefore \left|\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx)r^k - f(x)\right|$$

$$\leq \left|\left[\frac{a_0}{2} + \sum_{k=1}^{N-1}(a_k\cos kx + b_k\sin kx)r^k\right] - \left[\frac{a_0}{2} + \sum_{k=1}^{N-1}(a_k\cos kx + b_k\sin kx)\right]\right|$$

$$+ \left|\sum_{k=N}^{\infty}(a_k\cos kx + b_k\sin kx)r^k\right| + \left|\sum_{k=N}^{\infty}(a_k\cos kx + b_k\sin kx)\right|$$

$$\leq \sum_{k=1}^{N-1}|a_k\cos kx + b_k\sin kx||r^k-1| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$\leq \sum_{k=1}^{N-1}(|a_k|+|b_k|)(1-r^k) + \frac{2}{3}\varepsilon < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon \quad \text{if } \delta<r<1,\ x\in\mathbb{R}.$$

i.e. $\lim_{r\to 1-}\left(\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx)r^k\right) = f(x)$ uniformly on $\mathbb{R}$.

i.e. Sf is Abel summable to f uniformly on $\mathbb{R}$.

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#4(d) $a_k \geq 0 \Rightarrow 0 \leq S_n = \sum_{k=0}^{n} a_k \nearrow$ as $n \nearrow$.

Suppose that S_n is unbounded, i.e. $S_n \nearrow \infty$ as $n \longrightarrow \infty$.

Then for any $M > 0$, $\exists\, N \in \mathbb{N}$ s.t.

$$S_n > M \quad \text{if} \quad n \geq N.$$

$\sum_{k=0}^{\infty} a_k$ is Abel summable to $L \Rightarrow \sum_{k=0}^{\infty} a_k r^k$ converges $\forall r \in (0,1)$

$$\xrightarrow{B_6(a)} \sum_{k=0}^{\infty} a_k r^k = (1-r)\sum_{k=0}^{\infty} S_k r^k \geq (1-r)\sum_{k=N}^{\infty} S_k r^k > (1-r)\sum_{k=N}^{\infty} M r^k = M r^N$$

for each $r \in (0,1)$

$$\Rightarrow L = \lim_{r \to 1-} \sum_{k=0}^{\infty} a_k r^k \geq \lim_{r \to 1-} M r^N = M.$$

M arbitrary $\Rightarrow L = \infty$, a contradiction.

$\therefore$ S_n is bdd.

$\Rightarrow S_n$ converges to some $\tilde{L}$ as $n \to \infty$.

§7.4. Ex #7(a) $\Rightarrow \sum_{k=0}^{\infty} a_k$ is Abel summable to $\tilde{L}$.

$\Rightarrow L = \tilde{L}$

i.e. $\sum_{k=0}^{\infty} a_k = L$

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#5(c) f is Lipschitz of order α

$$\Rightarrow |f(x+h)-f(x-h)| \le |f(x+h)-f(x)| + |f(x-h)-f(x)|$$
$$\le M|h|^\alpha + M|-h|^\alpha = 2M|h|^\alpha = 2M\left(\frac{\pi}{2^{n+1}}\right)^\alpha.$$

By (a), $\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+h)-f(x-h)|^2\,dx = 4\sum_{k=0}^{\infty}\left(a_k^2(f)+b_k^2(f)\right)\sin^2 kh$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} 4M^2\left(\frac{\pi}{2^{n+1}}\right)^{2\alpha}dx \ge \ldots \ge 4\sum_{k=2^{n-1}}^{2^n-1}\left(a_k^2(f)+b_k^2(f)\right)\sin^2 kh$$

$$= 8M^2\left(\frac{\pi}{2^{n+1}}\right)^{2\alpha}, \qquad \ge \text{ by (b) } \quad 4\sum_{k=2^{n-1}}^{2^n-1}\left(a_k^2(f)+b_k^2(f)\right)\cdot\frac{1}{2}$$

$$\Rightarrow \sum_{k=2^{n-1}}^{2^n-1}\left(a_k^2(f)+b_k^2(f)\right) \le 4M^2\left(\frac{\pi}{2^{n+1}}\right)^{2\alpha} \quad \text{for } n\in\mathbb{N}$$

(d). $$\sum_{k=1}^{\infty}\left(|a_k(f)|+|b_k(f)|\right) = \sum_{n=1}^{\infty}\left(\sum_{k=2^{n-1}}^{2^n-1}\left(|a_k(f)|+|b_k(f)|\right)\right)$$
$$\le \sum_{n=1}^{\infty}\left\{2^{\frac{n}{2}}\left(\sum_{k=2^{n-1}}^{2^n-1}\left(a_k^2(f)+b_k^2(f)\right)\right)^{\frac12}\right\}$$
$$\overset{\text{by (c)}}{\le} \sum_{n=1}^{\infty}\left\{2^{\frac{n}{2}}\cdot 2M\left(\frac{\pi}{2^{n+1}}\right)^{\alpha}\right\}$$
$$= \sqrt{2}M\pi^\alpha \sum_{n=1}^{\infty}\left(\frac{1}{2^{\alpha-\frac12}}\right)^{n+1} < \infty \qquad \left(\because 0<\frac{1}{2^{\alpha-\frac12}}<1\right)$$

$$|a_k(f)\cos kx + b_k(f)\sin kx| \le |a_k(f)| + |b_k(f)| \qquad \forall x\in\mathbb{R}$$

Weierstrass M-test $\Rightarrow$ Sf converges absolutely & uniformly on $\mathbb{R}$.

(e) It's easy to see that f' is also periodic.

f' is conti. on $[-\pi,\pi]$ $\Rightarrow$ $|f'(x)|\le M$ $\forall x\in[-\pi,\pi]$ for some $M>0$.

$\Rightarrow |f'(x)|\le M$ $\forall x\in\mathbb{R}$.

Mean Value Theorem $\Rightarrow$ $|f(x+h)-f(x)| = |f'(c)||h| \le M|h|$ for some c between x & $x+h$, for $x,h\in\mathbb{R}$.

i.e. f is Lipschitz of order 1.

Then (d) $\Rightarrow$ Sf converges absolutely & uniformly on $\mathbb{R}$.

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#5 (a) For each $h \in \mathbb{R}$, define $g_h : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x+h) - f(x-h)$.

Then g_h is periodic, conti. (because f is conti.).

Parseval's identity $\Rightarrow$

$$\frac{|a_0(g_h)|^2}{2} + \sum_{k=1}^{\infty}\left(|a_k(g_h)|^2 + |b_k(g_h)|^2\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} |g_h(x)|^2\,dx$$

$$\| $$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+h) - f(x-h)|^2\,dx$$

To compute the Fourier coefficients of g_h:

$$a_0(g_h) = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x+h) - f(x-h))\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+h)\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-h)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi+h}^{\pi+h} f(x)\,dx - \frac{1}{\pi}\int_{-\pi-h}^{\pi-h} f(x)\,dx$$

$\because f$ is periodic, $= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$

$$a_k(g_h) = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x+h) - f(x-h))\cos kx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+h)\cos kx\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-h)\cos kx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi+h}^{\pi+h} f(x)\cos(kx - kh)\,dx - \frac{1}{\pi}\int_{-\pi-h}^{\pi-h} f(x)\cos(kx + kh)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx - kh)\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx + kh)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\left(\cos(kx - kh) - \cos(kx + kh)\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, 2\sin kx \sin kh\,dx$$

$$= 2\sin kh\; b_k(f)$$

Similarly, $b_k(g_h) = -2\sin kh\; a_k(f)$

$$\therefore \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+h) - f(x-h)|^2\,dx = 4\sum_{k=1}^{\infty}\left(a_k^2(f) + b_k^2(f)\right)\sin^2 kh.$$

(b) $k \in [2^{n-1}, 2^n]$, $h = \frac{\pi}{2^{n+1}} \Rightarrow kh \in [\frac{\pi}{4}, \frac{\pi}{2}]$

$\Rightarrow \sin kh \geq \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \Rightarrow \sin^2 kh \geq \frac{1}{2} \quad \forall k \in [2^{n-1}, 2^n]$

»

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#6. f is of b.v. on $[-\pi, \pi]$ $\Rightarrow$ f is the difference of two increasing functions on $[-\pi, \pi]$. Since the discontinuous pts of an increasing function is at most countable, the discontinuous pts of f on $[-\pi, \pi]$ is at most countable. $\Rightarrow$ The discontinuous pts of f on $\mathbb{R}$ is at most countable. A countable set is of measure zero $\Rightarrow$ f conti. except a set of measure zero. Dirichlet - Jordan Thm $\Rightarrow$ $S_N \longrightarrow f$ as $N \to \infty$ except a set of measure zero. i.e. $S_N f \longrightarrow f$ a.e. as $N \to \infty$.