§14.1

#5. (1) $f_N \longrightarrow f$ uniformly on $[-\pi, \pi]$ as $N \longrightarrow \infty$

$\therefore$ Given $\varepsilon > 0$, $\exists\ N_0 \in \mathbb{N}$ s.t.

$$|f_N(x) - f(x)| < \frac{\varepsilon}{3\pi} \quad \text{if} \quad N \geq N_0,\ x \in [-\pi, \pi].$$

$$\therefore \quad |a_k(f_N) - a_k(f)| \leq \int_{-\pi}^{\pi} |f_N(x) - f(x)||\cos kx|\, dx$$

$$\leq \int_{-\pi}^{\pi} \frac{\varepsilon}{3\pi}\, dx < \varepsilon. \quad \text{if} \quad N \geq N_0$$

$$|b_k(f_N) - b_k(f)| \leq \int_{-\pi}^{\pi} |f_N(x) - f(x)||\sin kx|\, dx$$

$$\leq \int_{-\pi}^{\pi} \frac{\varepsilon}{3\pi}\, dx < \varepsilon \quad \text{if} \quad N \geq N_0$$

$\therefore$ $a_k(f_N) \longrightarrow a_k(f)$, $b_k(f_N) \longrightarrow b_k(f)$ as $N \longrightarrow \infty$, uniformly in k.

(2). $$|a_k(f_N) - a_k(f)| \leq \int_{-\pi}^{\pi} |f_N(x) - f(x)|\, dx \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty$$

$$|b_k(f_N) - b_k(f)| \leq \int_{-\pi}^{\pi} |f_N(x) - f(x)|\, dx \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty$$

$\therefore$ $a_k(f_N) \longrightarrow a_k(f)$, $b_k(f_N) \longrightarrow b_k(f)$ as $N \longrightarrow \infty$, uniformly in k.

§14.2

#2. f is integrable on $[-\pi,\pi]$. $\therefore f$ is bdd. on $[-\pi,\pi]$.

i.e. $\exists M>0$ s.t. $|f(x)| \le M \quad \forall x \in [-\pi,\pi]$.

By Lemma 14.12.,

$$\begin{aligned}|(\sigma_N f)(x)| &= \frac{1}{\pi}\left|\int_{-\pi}^{\pi} f(x-t)K_N(t)\,dt\right| \\ &\le \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x-t)|\,K_N(t)\,dt \\ &\le \frac{M}{\pi}\int_{-\pi}^{\pi}K_N(t)\,dt \underset{\text{Lemma 14.13}}{=\!=\!=} M\end{aligned}$$

for all $x\in\mathbb{R}$, $N\in\mathbb{N}$.

#3. Let $S_N = \frac{a_0}{2}+\sum_{k=1}^{N}(a_k\cos kx + b_k\sin kx)$ for $N\in\mathbb{N}$, $S_0=\frac{a_0}{2}$,

then $\sigma_N(x) = \dfrac{S_0+S_1+\cdots\cdots+S_N}{N+1}$

$(\Rightarrow)$ It's obvious by Corollary 14.15

$(\Leftarrow)$. Assume that σ_N converges uniformly to f on $\mathbb{R}$.

σ_N is conti. and periodic $\Rightarrow$ f is conti. and periodic.

And by §14.1. Ex #5(a), $a_k(\sigma_N)\longrightarrow a_k(f)$, $b_k(\sigma_N)\longrightarrow b_k(f)$ as $N\to\infty$.

But for each k fixed, for $N\ge k$,

$$\begin{aligned}a_0(\sigma_N) &= a_0 \\ a_k(\sigma_N) &= \left(1-\frac{k}{N+1}\right)a_k \longrightarrow a_k \quad \text{as } N\longrightarrow\infty \\ b_k(\sigma_N) &= \left(1-\frac{k}{N+1}\right)b_k \longrightarrow b_k \quad \text{as } N\longrightarrow\infty\end{aligned}$$

$\therefore a_k(f) = a_k \quad \forall k\in\mathbb{N}\cup\{0\}$

$b_k(f) = b_k \quad \forall k\in\mathbb{N}$

i.e. S is the Fourier series of f.

§4.2

#5(a). Denote $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_1 x + a_0 = \sum_{k=0}^{n} a_k x^k$

$$\int_a^b P(x) f(x)\, dx = \sum_{k=0}^{n} a_k \int_a^b x^k f(x)\, dx = 0.$$

(b). Given $\varepsilon > 0$, by Weierstrass approximation Thm, $\exists$ a polynomial $P_\varepsilon(x)$ s.t.

$$|f(x) - P_\varepsilon(x)| < \varepsilon \qquad \forall x \in [a,b].$$

$$\therefore 0 \le \int_a^b |f(x)|^2 dx = \int_a^b f(x) \cdot \left(f(x) - P_\varepsilon(x) + P_\varepsilon(x) \right) dx$$

$$= \int_a^b f(x)\left(f(x) - P_\varepsilon(x)\right) dx + \int_a^b f(x) P_\varepsilon(x)\, dx$$

$$= \int_a^b f(x)\left(f(x) - P_\varepsilon(x)\right) dx \qquad \text{(by (a))}$$

$$\le \int_a^b |f(x)|\,|f(x) - P_\varepsilon(x)|\, dx$$

$$\le \varepsilon \int_a^b |f(x)|\, dx$$

ε arbitrary. $\therefore 0 \le \int_a^b |f(x)|^2 dx \le 0$

i.e. $\int_a^b |f(x)|^2 dx = 0.$

(c). f is conti. $\Rightarrow |f|^2$ is conti.

$\int_a^b |f(x)|^2 dx = 0$ & $|f|^2 \ge 0$, $|f|^2$ conti.

$\Rightarrow |f(x)|^2 = 0 \quad \forall x \in [a,b] \quad$ (by §5.1 Ex #4(b))

$\Rightarrow f(x) = 0 \quad \forall x \in [a,b].$

#8. f is Riemann integrable on $[-\pi, \pi]$

$\xRightarrow{\text{Lebesgue Thm}}$ f is conti. a.e. on $[-\pi,\pi]$.

(Note that $\lim_{h\to 0} \dfrac{f(x_0+h) + f(x_0-h)}{2} = f(x_0)$ if f is conti. at x_0)

$\therefore \sigma_N f \longrightarrow f$ as $N \to \infty$ for a.e. $x \in [-\pi, \pi]$.

f is periodic of period $2\pi \Rightarrow \sigma_N f \longrightarrow f$ as $N \to \infty$ for a.e. $x \in \mathbb{R}$.

§14.2

#4(a). Suppose that $(S_N f)(x_0) \longrightarrow \tilde{L}$ as $N \to \infty$ for some $\tilde{L} \in \mathbb{R}$.

By Rmk 14.11, the Cesáro mean $(\sigma_N f)(x_0) \longrightarrow \tilde{L}$ as $N \to \infty$

$\therefore L = \tilde{L}$.

i.e. $(S_N f)(x_0) \longrightarrow L$ as $N \to \infty$

(b). Let $f: \mathbb{R} \to \mathbb{R}$ be periodic of period 2π and

$$f(x) = \sqrt{2}\pi \cos\sqrt{2}x \quad \text{for} \quad x \in [-\pi, \pi].$$

Then f is conti on $\mathbb{R}$ since $f(-\pi) = f(\pi) = \sqrt{2}\pi\cos(\sqrt{2}\pi)$.

$a_0(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sqrt{2}\pi \cos\sqrt{2}x \, dx = \sin\sqrt{2}x \Big|_{-\pi}^{\pi} = 2\sin\sqrt{2}\pi$.

$a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sqrt{2}\pi \cos\sqrt{2}x \cos kx \, dx = \frac{4}{2-k^2}\sin\sqrt{2}\pi \cos k\pi = \frac{4(-1)^k}{2-k^2}\sin\sqrt{2}\pi$.

$b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sqrt{2}\pi \cos\sqrt{2}x \sin kx \, dx = 0$

Hence $(S_N f)(x) = \sin\sqrt{2}\pi + \sum_{k=1}^{N} \frac{4(-1)^k \sin\sqrt{2}\pi}{2-k^2}\cos kx$.

$\because$ $|\sin\sqrt{2}\pi| \le 1$, $\left| \frac{4(-1)^k \sin\sqrt{2}\pi}{2-k^2}\cos kx \right| \le \frac{4}{|2-k^2|} \le \frac{8}{k^2} \quad \forall x$

& $1 + \sum_{k=1}^{\infty} \frac{8}{k^2} < \infty$

$\xRightarrow{\text{Weierstrass M-test}}$ $S_N f$ converges uniformly on $\mathbb{R}$ as $N \to \infty$.

But by Corollary 14.15, $\sigma_N f$ converges to f uniformly on $\mathbb{R}$ as $N \to \infty$. $\therefore$ by (a), $S_N f$ converges to f uniformly on $\mathbb{R}$ as $N \to \infty$.

(Note that $S_N f$ converges to the periodic fcn f of period 2π, not the fcn $\sqrt{2}\pi\cos\sqrt{2}x$.)

§4.3

#5(d) f is even $\Longrightarrow$ $b_k = 0$ $\forall k \in \mathbb{N}$.

$\therefore (S_N f)(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos kx$.

But $|a_k \cos kx| \le |a_k|$ $\forall x$ & $\sum_{k=1}^{\infty} |a_k| < \infty$

$\xRightarrow{\text{Weierstrass M-test}}$ $S_N f$ converges uniformly and absolutely on $\mathbb{R}$ as $N \to \infty$.

f is not always continuous.

For example, $f(x) = \begin{cases} 1 & \text{if } x = 2k\pi \text{ for some } k \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$

Then f is periodic on $\mathbb{R}$, integrable on $[-\pi, \pi]$.

f is even & $a_0(f) = 0$, $a_k(f) = b_k(f) = 0$ $\forall k \in \mathbb{N}$.

But f is not continuous.

#6 (a). $\int_{-\pi}^{\pi} f(u + \frac{\pi}{k}) \cos ku \, du = \int_{-\pi+\frac{\pi}{k}}^{\pi+\frac{\pi}{k}} f(v) \cos(kv - \pi) \, dv$ $\left(\begin{array}{l} v = u + \frac{\pi}{k} \quad dv = du \\ ku = kv - \pi \end{array} \right)$

$= -\int_{-\pi+\frac{\pi}{k}}^{\pi+\frac{\pi}{k}} f(v) \cos kv \, dv = -\int_{-\pi}^{\pi} f(v) \cos kv \, dv$ ($\because f(v)\cos kv$ is periodic)

$\therefore \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(u) - f(u + \frac{\pi}{k})) \cos ku \, du = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(u) \cos ku + f(u) \cos ku) \, du$

$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \cos ku \, du = a_k(f)$.

(b). $\therefore |a_k(f)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(u) - f(u + \frac{\pi}{k})| |\cos ku| \, du$

$\le \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega(f, \frac{\pi}{k}) \, du = \omega(f, \frac{\pi}{k})$

Similarly, $b_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(u) - f(u + \frac{\pi}{k})) \sin ku \, du$ & $|b_k(f)| \le \omega(f, \frac{\pi}{k})$

(c). If f is periodic & conti., then f is uniformly conti.

$\Rightarrow \omega(f, \frac{\pi}{k}) \longrightarrow 0$ as $k \to \infty$.

$\Rightarrow a_k(f) \longrightarrow 0$, $b_k(f) \longrightarrow 0$ as $k \to \infty$ by (b).

§14.3

\#2. Suppose the contrary, i.e. $\exists\ f : [-\pi, \pi] \to \mathbb{R}$, conti. on $[-\pi,\pi]$ s.t. $|a_k(f)| \geq \frac{1}{\sqrt{k}}$ $\forall k \in \mathbb{N}$.

f is conti. on $[-\pi,\pi] \Rightarrow f$ is Riemann integrable on $[-\pi,\pi]$.

$\xrightarrow{\text{Thm 14.20}} \sum_{k=1}^{\infty} |a_k(f)|^2$ converges.

But $\sum_{k=1}^{\infty} |a_k(f)|^2 \geq \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}}\right)^2 = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$ $\times$.

$\therefore$ Such f doesn't exist.

\#5. $(S_N f)(x) = \frac{a_0}{2} + \sum_{l=1}^{N} (a_l \cos lx + b_l \sin lx)$ for $N \in \mathbb{N}$.

$\therefore (S_N f)(0) = \frac{a_0}{2} + \sum_{l=1}^{N} a_l$ $\quad (S_N f)(0) \geq 0$ $\forall N \in \mathbb{N} \cup \{0\}$.

(a). $(S_k f)(0) = \frac{a_0}{2} + \sum_{l=1}^{k} a_l \geq \frac{a_0}{2} + \sum_{l=1}^{j} a_l = (S_j f)(0)$ $\forall k \geq j \geq 0$

since $a_l \geq 0$ for $l = 0, 1, 2, \cdots$.

(b). $$(\sigma_{2N} f)(0) = \frac{(S_0 f)(0) + (S_1 f)(0) + \cdots + (S_{2N} f)(0)}{2N+1}$$

$$\geq \frac{(S_N f)(0) + (S_{N+1} f)(0) + \cdots + (S_{2N} f)(0)}{2N+1}$$

$$\geq \frac{(S_N f)(0) + (S_N f)(0) + \cdots + (S_N f)(0)}{2N+1} \quad \text{by (a)}$$

$$= \frac{(N+1)(S_N f)(0)}{2N+1} \geq \frac{1}{2}(S_N f)(0)$$

$\therefore 2(\sigma_{2N} f)(0) \geq (S_N f)(0)$.

(c). By §14.2 Ex#2, $\exists M > 0$ s.t. $|(\sigma_N f)(x)| \leq M$ $\forall x \in \mathbb{R}, N \in \mathbb{N}$.

In particular, $|(\sigma_N f)(0)| \leq M$ $\forall N \in \mathbb{N}$.

$\therefore (S_N f)(0) \leq 2(\sigma_{2N} f)(0) \leq 2M$ & $a_k \geq 0$ $\forall k \in \mathbb{N} \cup \{0\}$

$0 \leq \frac{a_0}{2} + \sum_{k=1}^{N} a_k$ $\left(= (S_N f)(0)\right)$

$\Rightarrow (S_N f)(0)$ converges i.e. $\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k < \infty$.

$\therefore \sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} a_k < \infty$