§12.3

Ex8 1°. First to show that Ω is a Jordan region in $\mathbb{R}^3$.

f is integrable on $E \Rightarrow f$ is bdd by some $M>0$ on E.

For any $\varepsilon>0$,

E is a Jordan region & f is integrable on E

$\Rightarrow \exists$ a rectangle $R=[a,b]\times[c,d]\subset\mathbb{R}^2$ and a grid $\mathcal{G}=\{R_1,R_2,\cdots,R_\ell\}$ on R s.t.

$$\sum_{\substack{R_j\in\mathcal{G}\\ R_j\cap\partial E\neq\phi}} |R_j| < \frac{\varepsilon}{3M}$$

$$\& \quad U(f,\mathcal{G})-L(f,\mathcal{G})<\frac{\varepsilon}{3}.$$

It's easy to see that

$$\begin{aligned}
\partial\Omega \subseteq\ & \{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in\partial E,\quad 0\le z\le f(x,y)\} \\
& \cup \{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in E,\quad z=f(x,y)\} \\
& \cup \{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in E,\quad z=0\} \\
\subseteq & \Bigg(\bigcup_{\substack{R_j\in\mathcal{G}\\ R_j\cap\partial E\neq\phi}} R_j\times[0,M]\Bigg) \cup \Bigg(\bigcup_{\substack{R_j\in\mathcal{G}\\ R_j\cap E\neq\phi}} R_j\times[m_j,M_j]\Bigg) \\
& \cup \Bigg(\bigcup_{\substack{R_j\in\mathcal{G}\\ R_j\cap E\neq\phi}} R_j\times[0,\frac{\varepsilon}{3|R|}]\Bigg)
\end{aligned}$$

Since

$$\begin{aligned}
& \sum_{\substack{R_j\in\mathcal{G}\\ R_j\cap\partial E\neq\phi}} |R_j\times[0,M]| + \sum_{\substack{R_j\in\mathcal{G}\\ R_j\cap E\neq\phi}} |R_j\times[m_j,M_j]| \\
& \quad + \sum_{\substack{R_j\in\mathcal{G}\\ R_j\cap E\neq\phi}} |R_j\times[0,\frac{\varepsilon}{3|R|}]| \\
= & \sum_{\substack{R_j\in\mathcal{G}\\ R_j\cap\partial E\neq\phi}} |R_j|\cdot M + \sum_{\substack{R_j\in\mathcal{G}\\ R_j\cap E\neq\phi}} |R_j|\cdot(M_j-m_j) + \sum_{\substack{R_j\in\mathcal{G}\\ R_j\cap E\neq\phi}} |R_j|\cdot\frac{\varepsilon}{3|R|} \\
< & \frac{\varepsilon}{3M}\cdot M + U(f,\mathcal{G})-L(f,\mathcal{G}) + |R|\cdot\frac{\varepsilon}{3|R|} \\
= & \varepsilon,
\end{aligned}$$

$\partial\Omega$ has volume zero.

$\therefore$ Ω is a Jordan region in $\mathbb{R}^3$.

2°. $\mathrm{Vol}(\Omega) = \iiint_\Omega dV$

$= \iiint_{R\times[0,M]} \chi_\Omega \, dV$ $\quad$ (By Thm 12.19, where $\chi_\Omega(x,y,z) = \begin{cases} 1 & \text{if } (x,y,z)\in\Omega \\ 0 & \text{otherwise} \end{cases}$)

$\underset{\ast}{=} \int_R \left(\int_0^M \chi_\Omega(x,y,z)\, dz \right) dA(x,y)$

(By Lemma 12.36, since for each $(x,y)\in R$,

$\chi_\Omega(x,y,z) = \chi_{[0,f(x,y)]}(z)$

$\equiv \begin{cases} 1 & \text{if } z\in[0,f(x,y)] \\ 0 & \text{otherwise.} \end{cases}$

is integrable on $[0,M]$)

$= \int_R \int_0^M \chi_{[0,f(x,y)]}(z) \, dz \, dA(x,y)$

$= \int_R f(x,y)\, dA(x,y)$.

§12.3

Ex10. For any $a<a'<b'<b$, $f: [a', b'] \times [c, d] \longrightarrow \mathbb{R}$ is conti.;
hence f is integrable on $[a', b'] \times [c, d]$ and $f(x, \cdot)$ is integrable on $[c, d]$ for each $x \in [a', b']$, $f(\cdot, y)$ is integrable on $[a', b']$ for each $y \in [c, d]$.

Fubini Thm $\Rightarrow$

$$\int_{a'}^{b'} \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_{a'}^{b'} f(x, y)\, dx\, dy$$

$F(y) = \int_a^b f(x, y) dx$ converges uniformly on $[c, d]$

For each $\varepsilon > 0$, $\exists\, a < \alpha < \beta < b$ s.t.

$$\left| \int_a^b f(x, y)\, dx - \int_{a'}^{b'} f(x, y) dx \right| < \frac{\varepsilon}{d-c}$$

for all $a < a' < \alpha < \beta < b' < b$, $y \in [c, d]$.

And by Thm 11.8, $F(y)$ is conti. on $[c, d]$, hence is integrable on $[c,d]$

$\therefore$ For any $a < a' < \alpha < \beta < b' < b$,

$$\left| \int_c^d \int_a^b f(x, y)\, dx\, dy - \int_c^d \int_{a'}^{b'} f(x, y)\, dx\, dy \right|$$
$$\leq \int_c^d \left| \int_a^b f(x, y)\, dx - \int_{a'}^{b'} f(x, y) dx \right| dy$$
$$\leq \int_c^d \frac{\varepsilon}{d-c}\, dy = \varepsilon$$

i.e. $$\lim_{\substack{a' \searrow a \\ b' \nearrow b}} \int_c^d \int_{a'}^{b'} f(x, y)\, dx\, dy = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

$$\|$$

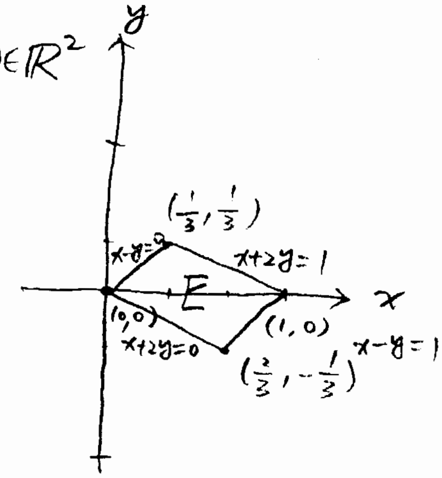
$$\lim_{\substack{a' \searrow a \\ b' \nearrow b}} \int_{a'}^{b'} \int_c^d f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dx\, dy.$$

§12.4

Ex5. (a). Let $\phi(x,y) = (x-y,\ x+2y) = (u,v)$ for $(x,y)\in\mathbb{R}^2$

$\phi \in C^1$, ϕ is 1-1, and

$$\Delta_\phi = \det\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = 3 \neq 0$$



$$\therefore \iint_E \sqrt{x-y}\sqrt{x+2y}\, dA$$

$$= \frac{1}{3}\int_0^1\int_0^1 \sqrt{u}\sqrt{v}\, du\, dv = \frac{4}{27}$$

(b). Let $\phi(x,y) = (x-3y,\ 2x+y) = (u,v)$ for $(x,y)\in\mathbb{R}^2$

$\phi\in C^1$, ϕ is 1-1 and

$$\Delta_\phi = \det\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} = 7 \neq 0$$

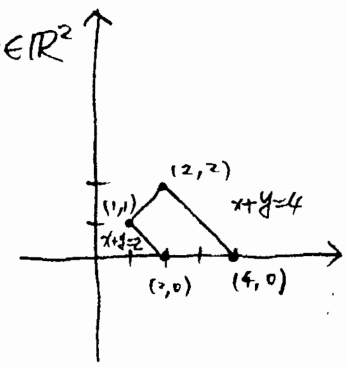
$$\therefore \iint_E \sqrt[3]{2x^2-5xy-3y^2}\, dA$$

$$= \frac{1}{7}\int_0^1\int_0^1 \sqrt[3]{uv}\, du\, dv = \frac{9}{112}$$

(c). Let $\phi(x,y) = (y+x,\ y-x) = (u,v)$ for $(x,y)\in\mathbb{R}^2$

$\phi\in C^1$, ϕ is 1-1 and

$$\Delta_\phi = \det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2 \neq 0$$

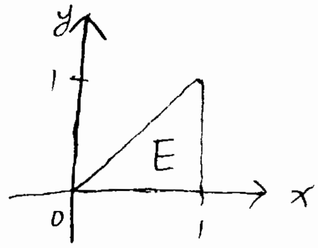


$$\iint_E e^{\frac{y-x}{y+x}}\, dA$$

$$= \frac{1}{2}\int_2^4\int_{-u}^0 e^{\frac{v}{u}}\, dv\, du = \frac{1}{2}\int_2^4 u\left(1-\frac{1}{e}\right) du = 3\left(1-\frac{1}{e}\right).$$

(d). Let $\phi(x,y) = (x-y,\ y) = (u,v)$ for $(x,y)\in\mathbb{R}^2$

ϕ is C^1, 1-1, and $\Delta_\phi = \det\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = 1 \neq 0$



$$\int_0^1\int_0^x f(x-y)\, dy\, dx = \iint_E f(x-y)\, dA$$

$$= \int_0^1\int_0^{1-u} f(u)\, dv\, du = \int_0^1 (1-u) f(u)\, du = 5$$

where $E = \{(x,y)\in\mathbb{R}^2 \mid 0\le y\le x\le 1\}$.

Ex6. By Lemma 11.40, $\exists R>0$ s.t. $\overline{B_r(x_0)} \subset V$, f is 1-1 on $B_r(x_0)$ $\forall 0<r<R$

Then Corollary 12.10 (iii) $\Rightarrow f(B_r(x_0))$, $f(\overline{B_r(x_0)})$ are Jordan regions if $0<r<R$.

Thm 11.39 $\Rightarrow f(B_r(x_0))$ is open if $0<r<R$.

$\overline{B_r(x_0)}$ is compact $\Rightarrow f(\overline{B_r(x_0)})$ is compact hence is closed.

$f(\overline{B_r(x_0)}) \setminus f(B_r(x_0)) \subset f(\partial B_r(x_0))$ which is of volume zero by Corollary 12.10 (i) $\Rightarrow f(\overline{B_r(x_0)}) \setminus f(B_r(x_0))$ is of volume zero. $\Rightarrow \mathrm{Vol}(f(B_r(x_0))) = \mathrm{Vol}(f(\overline{B_r(x_0)}))$.

$$\therefore \frac{\mathrm{Vol}(f(B_r(x_0)))}{\mathrm{Vol}(B_r(x_0))} = \frac{\mathrm{Vol}(f(\overline{B_r(x_0)}))}{\mathrm{Vol}(B_r(x_0))}$$

$$= \frac{1}{\mathrm{Vol}(B_r(x_0))} \int_{f(\overline{B_r(x_0)})} dx \quad \text{by Thm 12.22.}$$

$$= \frac{1}{\mathrm{Vol}(B_r(x_0))} \int_{\overline{B_r(x_0)}} |\Delta_f(x)|\, dx$$

(Note that 1 is integrable on $\overline{B_r(x_0)}$)

$$= \frac{1}{\mathrm{Vol}(B_r(x_0))} \int_{B_r(x_0)} |\Delta_f(x)|\, dx$$

$\longrightarrow |\Delta_f(x_0)|$ as $r \to 0$ by §12.2 Ex5.

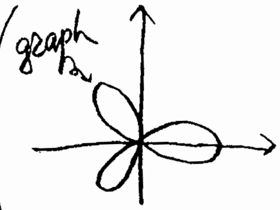
(Δ_f is conti. at x_0 since f is C^1.)

§13.1

Ex3. Consider $I = [0, 2\pi]$, $f: I \to \mathbb{R}$ defined by $f(\theta) = \cos 3\theta$ for $\theta \in I$. Then

$$|f(\theta)|^2 + |f'(\theta)|^2 = (\cos 3\theta)^2 + (-3\sin 3\theta)^2$$
$$= 1 + 8\sin^2 3\theta \neq 0 \qquad \forall \theta \in I.$$

But the graph of $r = f(\theta)$ is not a smooth C' curve in $\mathbb{R}$.

(

Any parametrization of the graph can't be 1-1 in the interior of the domain by the mediate value property of conti. fcns.)

Ex4 For each $K \in \mathbb{N}$, let $t_0 = 0$, $t_K = 1$, $t_k = \frac{1}{(K-k+\frac{1}{2})\pi}$ for $1 \le k \le K$.

Then $\{0 = t_0 < t_1 < t_2 < \cdots < t_K = 1\}$ is a partition of $[0,1]$,

and $\sum_{k=1}^{K} \| (t_k, f(t_k)) - (t_{k-1}, f(t_{k-1})) \|$

$\geq \sum_{k=1}^{K} |f(t_k) - f(t_{k-1})| = \sum_{k=1}^{K} \left| \sin(K-k+\frac{1}{2})\pi - \sin(K-k+\frac{3}{2})\pi \right|$

$= 2K \longrightarrow \infty$ as $K \longrightarrow \infty$.

$\therefore$ The curve $y = \sin(\frac{1}{x})$, $0 < x \le 1$ is not rectifiable.

(The domain of the parametrization of the graph is $(0,1]$, not a closed interval, hence the curve is not an arc)

Ex8 $\tau'(u) \ne 0$ for all but finitely many $u \in J$

$\Rightarrow$ J can be expressed as a finite union of disjoint intervals J_n, $1 \le n \le N$, such that $\tau'(u) \ne 0$ in $\mathring{J}_n$ for each $1 \le n \le N$.

$\therefore$ τ is strictly monotone in each J_n, $1 \le n \le N$

$\therefore \int_J g(\psi(u)) \|\psi'(u)\| \, du = \int_{\bigcup_{n=1}^{N} J_n} g(\psi(u)) \|\psi'(u)\| \, du$

$= \sum_{n=1}^{N} \int_{J_n} g(\psi(u)) \|\psi'(u)\| \, du$ ($\because$ J_n is disjoint)

$= \sum_{n=1}^{N} \int_{J_n} g(\phi(\tau(u))) \|\phi'(\tau(u))\| |\tau'(u)| \, du$

($\psi(u) = \phi(\tau(u)) \Rightarrow \psi'(u) = \phi'(\tau(u)) \, \tau'(u)$)

$= \sum_{n=1}^{N} \int_{\tau(J_n)} g(\phi(t)) \|\phi'(t)\| \, dt$ by Thm 5.34 (change of variables).

$= \int_{\bigcup_{n=1}^{N} \tau(J_n)} g(\phi(t)) \|\phi'(t)\| \, dt$ ($\because$ τ is 1-1 $\Rightarrow$ $\tau(J_n)$ disjoint)

$= \int_I g(\phi(t)) \|\phi'(t)\| \, dt$ (τ is onto $\Rightarrow$ $I = \bigcup_{n=1}^{N} \tau(J_n)$).

(The integrability of the integrals comes from the continuity of g)

§13.1

Ex10 (a) $\psi(t) = \mathbb{a} + t\mathbb{b}$ for $t \in I$

$\psi'(t) = \mathbb{b} \quad \forall t \in I$

For each $x_0 = \psi(t_0) \in \Lambda = \phi(I) \quad (t_0 \in I)$,

$\theta(t) =$ the angle between $\psi'(t) = \mathbb{b}$ & $\psi'(t_0) = \mathbb{b}$
$= 0$

$$\ell(t) = \left| \int_{t_0}^{t} \|\psi'(\tau)\| \, d\tau \right| = \|\mathbb{b}\| \, |t - t_0| > 0$$

$$\therefore \; K(x_0) = \lim_{t \to t_0} \frac{\theta(t)}{\ell(t)} = 0$$

(b) $\psi(t) = (r\cos t, \; r\sin t)$ for $t \in I$

$\psi'(t) = (-r\sin t, \; r\cos t)$

For each $x_0 = \psi(t_0) \in C \quad (t_0 \in I)$,

$\theta(t) =$ the angle between $(-r\sin t, r\cos t)$ & $(-r\sin t_0, r\cos t_0)$
$= |t - t_0|$

$$\ell(t) = \left| \int_{t_0}^{t} \|\psi'(\tau)\| \, d\tau \right| = r|t - t_0|$$

$$\therefore \; K(x_0) = \lim_{t \to t_0} \frac{|t - t_0|}{r|t - t_0|} = \frac{1}{r}$$

§13.1

Ex 11. (a) $s = s(t) = \ell(t) = \int_a^t \|\phi'(u)\|\,du \qquad t \in [a,b].$

$$\ell'(t) = \|\phi'(t)\| > 0 \qquad \forall t \in [a,b]$$

$\Rightarrow \ell^{-1}$ exists on $\ell([a,b]) = [0, L]$

$$\frac{d\ell^{-1}}{ds} = \frac{1}{\ell'(\ell^{-1}(s))} = \frac{1}{\|\phi'(\ell^{-1}(s))\|} \quad \text{on } [0,L].$$

$$\therefore\ \nu'(s) = \phi'(\ell^{-1}(s)) \cdot \frac{1}{\|\phi'(\ell^{-1}(s))\|} \quad \text{on } [0,L].$$

$$\Rightarrow \|\nu'(s)\| = \frac{\|\phi'(\ell^{-1}(s))\|}{\|\phi'(\ell^{-1}(s))\|} = 1 \qquad \forall s \in [0, L]$$

(b). $1 = \|\nu'(s)\|^2 = \nu'(s) \cdot \nu'(s) \qquad \forall s \in [0,L]$

$$\overset{\text{diff. w.r.t. } s}{\Rightarrow} 0 = 2\,\nu'(s) \cdot \nu''(s) \qquad \forall s \in [0,L]$$

i.e. $\nu'(s)$ & $\nu''(s)$ are orthogonal for each $s \in [0,L]$.

(c). Note that $\theta(s) \to 0$ as $s \to s_0$.

$$\therefore\ \frac{\sin\frac{\theta(s)}{2}}{\frac{\theta(s)}{2}} \to 1 \quad \text{as} \quad s \to s_0.$$

$$\therefore\ \lim_{s\to s_0} \frac{\theta(s)}{\ell(s)} = \lim_{s\to s_0} \frac{\frac{\theta(s)}{2}}{\sin\frac{\theta(s)}{2}} \frac{2\sin\frac{\theta(s)}{2}}{\ell(s)} = \lim_{s\to s_0} \frac{2\sin\frac{\theta(s)}{2}}{\ell(s)}$$

$$= \lim_{s\to s_0} \frac{\|\nu'(s) - \nu'(s_0)\|}{|s - s_0|} = \|\nu''(s_0)\|$$

$$\left(\begin{aligned} \because\ \|\nu'(s) - \nu'(s_0)\|^2 &= \|\nu'(s)\|^2 + \|\nu'(s_0)\|^2 - 2\|\nu'(s)\|\|\nu'(s_0)\|\cos\theta(s) \\ &= 2 - 2\cos\theta(s) = 4\sin^2\frac{\theta(s)}{2} \end{aligned} \right)$$

(d). By (c), $\kappa(x_0) = \|\nu''(s_0)\| = \|\nu''(s_0)\|\|\nu'(s_0)\|\sin\frac{\pi}{2}$

$$= \|\nu'(s_0) \times \nu''(s_0)\| \qquad (\because\ \nu'(s_0) \perp \nu''(s_0) \text{ by (b)})$$

$$\nu'(s) = \frac{\phi'(\ell^{-1}(s))}{\|\phi'(\ell^{-1}(s))\|}$$

$$\nu''(s) = \frac{\phi''(\ell^{-1}(s))}{\|\phi'(\ell^{-1}(s))\|} \cdot \frac{d\ell^{-1}(s)}{ds} - \frac{\phi'(\ell^{-1}(s))}{\|\phi'(\ell^{-1}(s))\|^3} \left(\phi'(\ell^{-1}(s)) \cdot \phi''(\ell^{-1}(s))\right) \cdot \frac{d\ell^{-1}(s)}{ds}$$

$$= \frac{\phi''(\ell^{-1}(s))}{\|\phi'(\ell^{-1}(s))\|^2} - \frac{\phi'(\ell^{-1}(s))\cdot\phi''(\ell^{-1}(s))}{\|\phi'(\ell^{-1}(s))\|^4}\phi'(\ell^{-1}(s))$$

$$\|\nu''(s)\|^2 = \nu''(s)\cdot\nu''(s)$$

$$= \frac{\|\phi''(\ell^{-1}(s))\|^2}{\|\phi'(\ell^{-1}(s))\|^4} + \frac{|\phi'(\ell^{-1}(s))\cdot\phi''(\ell^{-1}(s))|^2}{\|\phi'(\ell^{-1}(s))\|^8}\|\phi'(\ell^{-1}(s))\|^2$$

$$-2\frac{(\phi'(\ell^{-1}(s))\cdot\phi''(\ell^{-1}(s)))^2}{\|\phi'(\ell^{-1}(s))\|^6}$$

$$= \frac{\|\phi''(\ell^{-1}(s))\|^2\|\phi'(\ell^{-1}(s))\|^2 - (\phi'(\ell^{-1}(s))\cdot\phi''(\ell^{-1}(s)))^2}{\|\phi'(\ell^{-1}(s))\|^6}$$

$$\overset{(\text{in } \mathbb{R}^3)}{=} \frac{\|\phi'(\ell^{-1}(s))\times\phi''(\ell^{-1}(s))\|^2}{\|\phi'(\ell^{-1}(s))\|^6}$$

$$\therefore \underset{\kappa(x_0)}{\underset{\|}{\|\nu''(s_0)\|}} = \frac{\|\phi'(t_0)\times\phi''(t_0)\|}{\|\phi'(t_0)\|^3}$$

(e) Let $\phi(x) = (x, f(x))$

$\phi'(x) = (1, f'(x))$ $\quad\phi''(x) = (0, f''(x))$

By computation in (d),

$$\|\nu''(s)\|^2 = \frac{\|\phi''(\ell^{-1}(s))\|^2\|\phi'(\ell^{-1}(s))\|^2 - (\phi'(\ell^{-1}(s))\cdot\phi''(\ell^{-1}(s)))^2}{\|\phi'(\ell^{-1}(s))\|^6}$$

$$= \frac{f''(\ell^{-1}(s))^2(1+f'(\ell^{-1}(s))^2) - (f'(\ell^{-1}(s))f''(\ell^{-1}(s)))^2}{(1+f'(\ell^{-1}(s))^2)^3}$$

$$= \frac{f''(\ell^{-1}(s))^2}{(1+f'(\ell^{-1}(s))^2)^3}$$

$$\therefore \kappa(x_0, y_0) = \frac{|f''(x_0)|}{(1+f'(x_0)^2)^{3/2}}$$

§13.2

Ex2 (a) $\phi(t)=(t,t^2)$ $\quad t\in[1,3]$ $\quad \phi'(t)=(1,2t)$

$$\int_C F\cdot T\,ds=\int_1^3 (t\cdot t^2,\ t^2-t)\cdot(1,2t)\,dt$$
$$=\int_1^3(3t^3-2t^2)\,dt=\frac{128}{3}$$

(b) $\begin{cases} y^2+2z^2=1\\ x=-1\end{cases}$

Let $\phi(t)=(-1,\ \cos t,\ \frac{1}{\sqrt2}\sin t)$ $\quad t\in[0,2\pi]$

$$\phi'(t)=(0,\ -\sin t,\ \tfrac{1}{\sqrt2}\cos t)$$

$$\int_C F\cdot T\,ds=\int_0^{2\pi}\left(\sqrt{(-1)^3+\cos^3t+5},\ \tfrac{1}{\sqrt2}\sin t,\ (-1)^2\right)\cdot\left(0,-\sin t,\tfrac{1}{\sqrt2}\cos t\right)dt$$
$$=\frac{1}{\sqrt2}\int_0^{2\pi}(-\sin^2t+\cos t)\,dt=-\frac{\pi}{\sqrt2}$$

(c) $\begin{cases} y=|x|\\ x^2+3z^2=1\end{cases}$

Let $\phi(t)=\left(\cos(-t),\ |\cos(-t)|,\ \frac{1}{\sqrt3}\sin(-t)\right)$ $\quad t\in[-\frac{\pi}{2},\frac{3}{2}\pi]$

$$=\left(\cos t,\ |\cos t|,\ -\tfrac{1}{\sqrt3}\sin t\right)$$

$$\phi'(t)=\begin{cases}\left(-\sin t,\ -\sin t,\ -\tfrac{1}{\sqrt3}\cos t\right) & t\in(-\frac{\pi}{2},\frac{\pi}{2})\\ \left(-\sin t,\ \sin t,\ -\tfrac{1}{\sqrt3}\cos t\right) & t\in(\frac{\pi}{2},\frac{3\pi}{2})\\ \text{not exists} & t=\frac{\pi}{2}\end{cases}$$

$$\int_C F\cdot T\,ds=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(-\tfrac{1}{\sqrt3}\sin t,\ \tfrac{1}{\sqrt3}\sin t,\ \cos t+|\cos t|\right)\cdot\left(-\sin t,-\sin t,-\tfrac{1}{\sqrt3}\cos t\right)dt$$
$$+\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\left(-\tfrac{1}{\sqrt3}\sin t,\ \tfrac{1}{\sqrt3}\sin t,\ \cos t+|\cos t|\right)\cdot\left(-\sin t,\ \sin t,\ -\tfrac{1}{\sqrt3}\cos t\right)dt$$
$$=-\frac{2}{\sqrt3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2t\,dt+\frac{2}{\sqrt3}\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\sin^2t\,dt=0$$

§13.2

Ex 4 (a). τ is 1-1, onto from $J=\tau^{-1}(I)$ to I.

$\tau'(u)=\delta>0 \quad \forall u\in\mathbb{R}$.

$\psi'=\phi'\tau'\neq 0$ in J.

$\therefore$ (ψ, J) is a smooth parametrization, and (ψ, J) is orientation equivalent to (ϕ, I).

(b). (ϕ, I) is a parametrization of some smooth arc

$\Rightarrow I=[a,b]$ for some $a,b\in\mathbb{R}$.

Let $\delta=b-a>0$, $c=a$ in (a), i.e. $\tau(u)=(b-a)u+a$.

Then τ^{-1} maps $I=[a,b]$ onto $J=[0,1]$.

By (a), $(\psi,[0,1])$ is orientation equivalent to (ϕ, I).

(c). If $\bigcup_{j=1}^{N}(\phi_j, I_j)$ is a parametrization of a piecewise smooth curve, then there exist functions ψ and τ on $[0,1]$ that are C' on $(0,1)\setminus\{\frac{j}{N}: j=1,\cdots,N\}$ such that $\tau'>0$ and $\psi=\phi_j\circ\tau$ on $(\frac{j-1}{N},\frac{j}{N})$ for each $j=1,\dots,N$.

pf: Easy! Similar to (b).

§13.2

Ex 7. (a) For $(x_1, y) \in V$, V is open

$\therefore$ The whole line segment between (x_1, y) & (x, y) lies in V if x closed to x_1 sufficiently.

For such x_1,

let $\phi(t) = (x_1 + t(x - x_1), y)$ $\quad t \in [0, 1]$

$\phi'(t) = (x - x_1, 0)$.

$$\int_{C(x)} F \cdot T \, ds = \int_0^1 (P(\phi(t)), Q(\phi(t))) \cdot (x - x_1, 0) \, dt$$

$$= (x - x_1) \int_0^1 P(x_1 + t(x - x_1), y) \, dt$$

$$= \int_{x_1}^{x} P(s, y) \, ds \qquad \begin{aligned} s &= x_1 + t(x - x_1) \\ ds &= (x - x_1) dt \end{aligned}$$

P is conti. $\Rightarrow$

$$\frac{\partial}{\partial x} \int_{C(x)} F \cdot T \, ds = \frac{\partial}{\partial x} \int_{x_1}^{x} P(s, y) \, ds = P(x, y)$$

Similarly, for $(x, y_1) \in V$, the whole line segment between (x, y_1) & (x, y) lies in V if y closed to y_1 sufficiently.

Let $C(y)$ be the vertical line segment oriented from (x, y_1) to (x, y).

Let $\psi(t) = (x, y_1 + t(y - y_1))$ $\quad t \in [0, 1]$

$\psi'(t) = (0, y - y_1)$

$$\int_{C(y)} F \cdot T \, ds = \int_0^1 (P(\psi(t)), Q(\psi(t))) \cdot (0, y - y_1) \, dt$$

$$= \int_0^1 (y - y_1) Q(x, y_1 + t(y - y_1)) \, dt$$

$$= \int_{y_1}^{y} Q(x, s) \, ds$$

$$\frac{\partial}{\partial y} \int_{C(y)} F \cdot T \, ds = Q(x, y)$$

Ex 7 (b) ($\Rightarrow$). Suppose $C_1(x,y)$, $C_2(x,y)$ are two piecewise smooth curves start at (x_0,y_0), end at (x,y), and stay inside V. Denote the parametrization of $C_i(x,y)$ is $\bigcup_{j=1}^{N_i}(\phi_j^{(i)}, I_j^{(i)})$ for $i=1,2$. First assume that $C_1(x,y)$ & $C_2(x,y)$ are disjoint. Consider the parametrization $\bigcup_{j=1}^{N_1}(\phi_j^{(1)}, I_j^{(1)}) \cup \bigcup_{j=1}^{N_2}(\phi_j^{(2)}, I_j^{(2)})$ of the curve $C(x,y) = C_1(x,y) - C_2(x,y)$. Obviously $C(x,y)$ is a closed piecewise smooth curve.

$$\therefore \int_{C(x,y)} F\cdot T\, ds = 0$$

$$= \int_{C_1(x,y)} F\cdot T\, ds - \int_{C_2(x,y)} F\cdot T\, ds$$

(just by the definition of line integral)

$$\Rightarrow \int_{C_1(x,y)} F\cdot T\, ds = \int_{C_2(x,y)} F\cdot T\, ds.$$

If $C_1(x,y)$ intersects $C_2(x,y)$ before (x,y), split $C_1(x,y)$ & $C_2(x,y)$ at the intersection points. Then apply the previous result to each parts to get the same result.

($\Leftarrow$). If $C(x,y)$ is a closed piecewise smooth curve in V. Assume $C(x,y)$ has a parametrization $\bigcup_{j=1}^{N}(\phi_j, I_j)$. Split I_N into two intervals $I_{N,1}$, $I_{N,2}$. Consider

$$\bigcup_{j=1}^{N-1}(\phi_j, I_j) \cup (\phi_N|_{I_{N,1}}, I_{N,1}) \ \& \ (-\phi_N|_{I_{N,2}}, I_{N,2})$$

parametrizations of $C_1(x,y)$ & $C_2(x,y)$. $C_1(x,y)$ and $C_2(x,y)$ start and end at the same points.

$$\therefore \int_{C_1(x,y)} F\cdot T\, ds = \int_{C_2(x,y)} F\cdot T\, ds$$

$$\Rightarrow \int_{C(x,y)} F\cdot T\, ds = \int_{C_1(x,y)} F\cdot T\, ds - \int_{C_2(x,y)} F\cdot T\, ds = 0.$$

(just by definition)

§13.2
Ex7(c) ($\Leftarrow$) By considering each connected component of V seperately, we may assume that V is connected.

V is open, connected $\Rightarrow$ V is polygonally connected by §10.5 Ex10(c)
i.e. for $(x_1, y_1), (x_2, y_2) \in V$, there is a polygonal path (which is piecewise smooth) connecting (x_1, y_1) & (x_2, y_2).

Fix $(x_0, y_0) \in V$, define $f: V \to \mathbb{R}$ as in (b), i.e.

$$f(x,y) = \int_{C(x,y)} F \cdot T \, ds$$

where $C(x,y)$ is a piecewise smooth curve starts at (x_0, y_0) ends at (x,y), stays inside V. The definition is well-defined by (b) and the extence of such curve $C(x,y)$.

Claim: $F(x,y) = \nabla f(x,y)$ in V.

For $(x,y) \in V$,

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h} = \lim_{h\to 0} \frac{1}{h} \int_{L((x,y),(x+h,y))} F \cdot T \, ds$$
$$= P(x,y) \quad \text{by (a)}$$

Similarly, $f_y(x,y) = Q(x,y)$

$\therefore F(x,y) = \nabla f(x,y)$ in V.

($\Rightarrow$) Suppose C is a closed piecewise smooth curve with parametrization $\bigcup_{j=1}^{N} (\phi_j, I_j)$ where $I_j = [a_j, b_j]$, $\phi_j(b_j) = \phi_{j+1}(a_{j+1})$, $\phi_1(a_1) = \phi_N(b_N)$

$$\int_C F \cdot T \, ds = \sum_{j=1}^{N} \int_{a_j}^{b_j} f_x(\phi_j(t)) \phi_{j,1}'(t) + f_y(\phi_j(t)) \phi_{j,2}'(t) \, dt$$
$$= \sum_{j=1}^{N} \int_{a_j}^{b_j} \frac{d}{dt} f(\phi_j(t)) \, dt = \sum_{j=1}^{N} f(\phi_j(b_j)) - f(\phi_j(a_j))$$
$$= f(\phi_N(b_N)) - f(\phi_1(a_1)) = 0 .$$

§13.2

Ex7(d). By (c), $\exists f : V \to \mathbb{R}$ s.t $F(x,y) = \nabla f(x,y)$ in V.

F is $C^1 \Rightarrow f$ is $C^2 \Rightarrow f_{xy} = f_{yx}$ in V

i.e. $\underbrace{\frac{\partial}{\partial x}\frac{\partial f}{\partial y}}_{\frac{\partial}{\partial x} Q} = \underbrace{\frac{\partial}{\partial y}\frac{\partial f}{\partial x}}_{\frac{\partial}{\partial y} P}$ in V

§13.3

Ex4. Let $\phi(u,v)=(u,v,0)$ for $(u,v)\in E$.

$$D\phi(u,v)=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$$

$$N_\phi(u,v)=(0,0,1) \qquad \|N_\phi(u,v)\|=1$$

$\therefore (\phi, E)$ is a smooth parametrization of S.

$$\iint_S d\sigma=\iint_E \|N_\phi(u,v)\|\,d(u,v)=\iint_E d(u,v)=\text{Area}(E).$$

$$\iint_S g\,d\sigma=\iint_E g(\phi(u,v))\,\|N_\phi(u,v)\|\,d(u,v)$$
$$=\iint_E g(u,v,0)\,d(u,v).$$

for each conti. $g: E\to\mathbb{R}$.

Ex5 (a). Obviously, (ψ, B) & (ϕ, E) are parametrizations of the same C^p surface. By Thm 13.36,

$$N_\psi(s,t)=\Delta_\tau(s,t)\,N_\phi(\tau(s,t))$$

for each $(s,t)\in B$.

$$\therefore \iint_B g(\psi(s,t))\,\|N_\psi(s,t)\|\,d(s,t)$$
$$=\iint_B g(\phi(\tau(s,t)))\,\|N_\phi(\tau(s,t))\|\,\Delta_\tau(s,t)\,d(s,t)$$
$$=\iint_E g(\phi(u,v))\,\|N_\phi(u,v)\|\,d(u,v) \qquad (E=\tau(B))$$

by Thm 12.46 (The integrability of the integrals just from the continuity of the functions.)

for all conti. $g: \phi(E)\to\mathbb{R}$.

Ex 8. $\sqrt{E^2G^2-F^2}=\sqrt{\|\psi_u\|^2\|\psi_v\|^2-(\psi_u\cdot\psi_v)^2}=\|\psi_u\times\psi_v\|=\|N_\psi\|$

$\therefore \int_B \sqrt{E^2G^2-F^2}\,d(u,v)=\int_B \|N_\psi(u,v)\|\,d(u,v)=\sigma(S).$

§13.4

Ex5. ∂E consists of 4 smooth pieces.

S_1: $x=0$, $y\geq 0$, $z\geq 0$, $y+z\leq 1$

S_2: $x\geq 0$, $y=0$, $z\geq 0$, $x+z\leq 1$

S_3: $x\geq 0$, $y\geq 0$, $z=0$, $x+y\leq 1$

S_4: $x\geq 0$, $y\geq 0$, $z\geq 0$, $x+y+z=1$.

On S_1: let $\phi(v,w)=(0,v,w)$ for $(v,w)\in\tilde{S}=\left\{(a,b)\,\middle|\,\begin{matrix}a\geq 0,\ b\geq 0\\ a+b\leq 1\end{matrix}\right\}$

$$D\phi(v,w)=\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}$$

$$N_\phi(v,w)=(1,0,0).$$

The outward normal of ∂E on S_1 is $(-1,0,0)$

$$\therefore \iint_{S_1} P\,dy\,dz+Q\,dz\,dx+R\,dx\,dy$$

$$=-\iint_{\tilde{S}} P(0,v,w)\,d(v,w)$$

Similarly,

$$\iint_{S_2} P\,dy\,dz+Q\,dz\,dx+R\,dx\,dy=-\iint_{\tilde{S}} Q(u,0,w)\,d(u,w)$$

$$\iint_{S_3} P\,dy\,dz+Q\,dz\,dx+R\,dx\,dy=-\iint_{\tilde{S}} R(u,v,0)\,d(u,v)$$

On S_4: let $\psi(u,v)=(u,v,1-u-v)$ for $(u,v)\in\tilde{S}$

$$D\psi(u,v)=\begin{pmatrix}1&0&-1\\0&1&-1\end{pmatrix}$$

$$N_\psi(u,v)=(1,1,1)$$

The outward normal of ∂E on S_4 is $(1,1,1)$

$$\therefore \iint_{S_4} P\,dy\,dz+Q\,dz\,dx+R\,dx\,dy$$

$$=\iint_{\tilde{S}} P(u,v,1-u-v)+Q(u,v,1-u-v)+R(u,v,1-u-v)\,d(u,v)$$

$$=\iint_{\tilde{S}} P(1-v-w,v,w)\,d(v,w)+\iint_{\tilde{S}} Q(u,1-u-w,w)\,d(u,w)$$

$$+\iint_{\tilde{S}} R(u,v,1-u-v)\,d(u,v).$$

$$\therefore \iint_{\partial E} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$

$$= \iint_{\tilde{S}} P(1-v-w, v, w) - P(0, v, w)\, d(v, w)$$

$$+ \iint_{\tilde{S}} Q(u, 1-u-w, w) - Q(u, 0, w)\, d(u, w)$$

$$+ \iint_{\tilde{S}} R(u, v, 1-u-v) - R(u, v, 0)\, d(u, v)$$

$$= \iint_{\tilde{S}} \int_0^{1-v-w} P_x(u, v, w)\, du \quad d(v, w)$$

$$+ \iint_{\tilde{S}} \int_0^{1-u-w} Q_y(u, v, w)\, dv \quad d(u, w)$$

$$+ \iint_{\tilde{S}} \int_0^{1-u-v} R_z(u, v, w)\, dz \quad d(u, v)$$

$$= \iiint_E P_x\, dV + \iiint_E Q_y\, dV + \iiint_E R_z\, dV$$

$$= \iiint_E (P_x + Q_y + R_z)\, dV.$$

§13.4

Ex6. $S = \bigcup_{j=1}^{3} S_j$

where $S_1: \quad x=0, \ y\geq 0, \ z\geq 0, \ y+z\leq 1$

$S_2: \quad x\geq 0, \ y=0, \ z\geq 0, \ x+z\leq 1$

$S_3: \quad x\geq 0, \ y\geq 0, \ z=0, \ x+y\leq 1.$

On S_1: let $\phi(v,w) = (0,v,w)$ for $(v,w)\in \tilde{S} = \{(a,b) \mid a\geq 0, b\geq 0, a+b\leq 1\}$.

$$D\phi(v,w) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$N_\phi(v,w) = (1,0,0)$$

The outward normal on S is $(-1,0,0)$,

$$\therefore \iint_{S_1} (R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dz\,dx + (Q_x - P_y)\,dx\,dy$$

$$= -\iint_{\tilde{S}} (R_y - Q_z)(0,v,w)\,d(v,w)$$

$$= -\int_0^1 \int_0^{1-w} R_y(0,v,w)\,dv\,dw - \int_0^1 \int_0^{1-v} Q_z(0,v,w)\,dw\,dv$$

$$= -\int_0^1 \big(R(0,1-w,w) - R(0,0,w)\big)dw + \int_0^1 \big(Q(0,v,1-v) - Q(0,v,0)\big)dv$$

Similarly, we can get

$$\iint_{S_2} (R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dz\,dx + (Q_x - P_y)\,dx\,dy$$

$$= -\int_0^1 \big(P(u,0,1-u) - P(u,0,0)\big)du + \int_0^1 \big(R(1-w,0,w) - R(0,0,w)\big)dw$$

$$\iint_{S_3} (R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dz\,dx + (Q_x - P_y)\,dx\,dy$$

$$= -\int_0^1 \big(Q(1-v,v,0) - Q(0,v,0)\big)dy + \int_0^1 \big(P(u,1-u,0) - P(u,0,0)\big)du$$

$$\therefore \iint_S (R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dz\,dx + (Q_x - P_y)\,dx\,dy$$

$$= \int_0^1 \big(-P(u,0,1-u) + P(u,1-u,0)\big)du + \int_0^1 \big(-Q(1-v,v,0) + Q(0,v,1-v)\big)dv$$

$$+ \int_0^1 \big(-R(0,1-w,w) + R(1-w,0,w)\big)dw$$

∂S consists of 3 smooth curves C_1, C_2 & C_3.

$C_1: \ x=0, \ y\geq 0, \ z\geq 0, \ y+z=1$, from $(0,0,1)$ to $(0,1,0)$

$C_2: \ x\geq 0, \ y=0, \ z\geq 0, \ x+z=1$, from $(1,0,0)$ to $(0,0,1)$

$C_3: \ x\geq 0, \ y\geq 0, \ z=0, \ x+y=1$ from $(0,1,0)$ to $(1,0,0)$

Along C_1: let $\psi(t) = (0, t, 1-t)$ for $t \in [0,1]$

$$\psi'(t) = (0, 1, -1)$$

$$\therefore \int_{C_1} P\,dx + Q\,dy + R\,dz$$

$$= \int_0^1 \big(Q(0, t, 1-t) - R(0, t, 1-t)\big)\,dt$$

$$= \int_0^1 \big(Q(0, t, 1-t) - R(0, 1-t, t)\big)\,dt$$

Similarly, $\iint_{C_2} P\,dx + Q\,dy + R\,dz$

$$= \int_0^1 \big(-P(t, 0, 1-t) + R(1-t, 0, t)\big)\,dt$$

$$\int_{C_3} P\,dx + Q\,dy + R\,dz$$

$$= \int_0^1 \big(P(t, 1-t, 0) - Q(1-t, t, 0)\big)\,dt$$

$$\therefore \iint_{\partial S} P\,dx + Q\,dy + R\,dz$$

$$= \int_0^1 \{P(t, 1-t, 0) - P(t, 0, 1-t) + Q(0, t, 1-t) - Q(1-t, t, 0)$$
$$+ R(1-t, 0, t) - R(0, 1-t, t)\}\,dt$$

$$= \iint_S (R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dz\,dx + (Q_x - P_y)\,dx\,dy$$

§13.5

Ex 5 (a). By Green's Thm (Thm 13.50) with $F(x,y)=(-\frac{1}{2}y, \frac{1}{2}x)$,

$$\frac{1}{2}\int_{\partial E} x\,dy - y\,dx = \iint_{\partial E} F\cdot T\,ds$$
$$= \iint_E \frac{\partial}{\partial x}(\tfrac{1}{2}x) - \frac{\partial}{\partial y}(-\tfrac{1}{2}y)\,dA = \iint_E \tfrac{1}{2}+\tfrac{1}{2}\,dA = \iint_E dA = \mathrm{Area}(E).$$

(b). $\phi(t)=\left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right)$, $t\in[0,\infty)$

$$\phi'(t) = \left(\frac{3-6t^3}{(1+t^3)^2}, \frac{6t-3t^4}{(1+t^3)^2}\right)$$

By (a),

$$\mathrm{Area} = \frac{1}{2}\int_{\partial E} x\,dy - y\,dx$$
$$= \frac{1}{2}\int_0^\infty \frac{3t}{1+t^3}\cdot\frac{6t-3t^4}{(1+t^3)^2} - \frac{3t^2}{1+t^3}\cdot\frac{3-6t^3}{(1+t^3)^2}\,dt$$
$$= \frac{1}{2}\int_0^\infty \frac{9t^2}{(1+t^3)^2}\,dt = -\frac{3}{2}\left.\frac{1}{1+t^3}\right|_0^\infty = \frac{3}{2}.$$

(c). If E is a Jordan region whose topological boundary is piecewise smooth surface oriented positively, then

$$\mathrm{Vol}(E) = \frac{1}{3}\iint_{\partial E} x\,dy\,dz + y\,dz\,dx + z\,dx\,dy.$$

(Just by Gauss's Thm (Thm 13.54) with $F(x,y,z)=(\frac{1}{3}x, \frac{1}{3}y, \frac{1}{3}z)$,

$$\frac{1}{3}\iint_{\partial E} x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \iint_{\partial E} F\cdot n\,d\sigma$$
$$= \iiint_E \operatorname{div} F\,dV = \iiint_E \left(\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{3}\right)dV = \iiint_E dV = \mathrm{Vol}(E))$$

(d). As in Example 13.32, let $\phi(u,v) = ((a+b\cos v)\cos u, (a+b\cos v)\sin u, b\sin v)$ for $(u,v)\in E=[-\pi,\pi]\times[-\pi,\pi]$.

$$D\phi(u,v) = \begin{pmatrix} (a+b\cos v)(-\sin u) & (a+b\cos v)\cos u & 0 \\ -b\sin v\cos u & -b\sin v\sin u & b\cos v \end{pmatrix}$$

$$N_\phi(u,v) = ((a+b\cos v)b\cos v\cos u, (a+b\cos v)b\cos v\sin u, (a+b\cos v)b\sin v)$$

$$\mathrm{Volume} = \frac{1}{3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (a+b\cos v)^2 b\cos v\cos^2 u + (a+b\cos v)^2 b\cos v\sin^2 u + (a+b\cos v)b^2\sin^2 v\,du\,dv$$
$$= \frac{1}{3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} ab^2 + ab^2\cos^2 v + (a^2b+b^3)\cos v\,du\,dv$$
$$= 2\pi^2 ab^2$$

§13.5

Ex6(a). Let $P = \frac{y}{x^2+y^2}$, $Q = \frac{-x}{x^2+y^2}$ for $(x,y) \in E = B_1(0,0)$

P, Q are conti. except $(0,0)$.

$$\int_{\partial E} F \cdot T\, ds = \int_0^{2\pi} \sin\theta(-\sin\theta) - \cos\theta(\cos\theta)\, d\theta = -\int_0^{2\pi} d\theta = -2\pi$$

But $\iint_E \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_E 0\, dA = 0 \neq -2\pi$.

$\therefore$ Green's Thm does not hold if continuity of P, Q is relaxed at one point in E.

(b). Consider $P = \frac{x}{(x^2+y^2+z^2)^{3/2}}$, $Q = \frac{y}{(x^2+y^2+z^2)^{3/2}}$, $R = \frac{z}{(x^2+y^2+z^2)^{3/2}}$ for $(x,y,z) \in E = B_1(0,0,0)$. P, Q, R are conti. except $(0,0,0)$

$$\iint_{\partial E} F \cdot n\, d\sigma = \int_0^{\pi}\int_0^{2\pi} \sin\varphi\, d\theta\, d\varphi = 4\pi$$

But $\operatorname{div}(P,Q,R) = 0$

$$\therefore \iiint_E \operatorname{div} F\, dV = 0 \neq 4\pi.$$

$\therefore$ Gauss's Thm does not hold if continuity of F is relaxed at one point in E.

§13.5

Ex9 (a) f is C^2 at $x_0 \Rightarrow f_{xy}(x_0) = f_{yx}(x_0)$

$$f_{xz}(x_0) = f_{zx}(x_0)$$
$$f_{yz}(x_0) = f_{zy}(x_0)$$

$$\nabla f(x_0) = (f_x(x_0), f_y(x_0), f_z(x_0))$$

$$\text{curl } \nabla f(x_0) = \left(\frac{\partial}{\partial y} f_z - \frac{\partial}{\partial z} f_y, \frac{\partial}{\partial z} f_x - \frac{\partial}{\partial x} f_z, \frac{\partial}{\partial x} f_y - \frac{\partial}{\partial y} f_x \right)(x_0)$$
$$= (f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy})(x_0) = (0, 0, 0).$$

(b). $\text{curl } F = (R_y - Q_z, P_z - R_x, Q_x - P_y)$ on E.

$$\text{div curl } F(x_0) = \left(\frac{\partial}{\partial x}(R_y - Q_z) + \frac{\partial}{\partial y}(P_z - R_x) + \frac{\partial}{\partial z}(Q_x - P_y) \right)(x_0)$$
$$= (R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz})(x_0) = 0.$$

(c). Apply Gauss's Thm, $\quad (f$ is $C^2 \Rightarrow F = \nabla f$ is $C^1)$

$$\iint_{\partial E} fF \cdot n \, d\sigma = \iiint_E (\text{div } fF) \, dV$$

But $\text{div } fF = \text{div}(f f_x, f f_y, f f_z)$

$$= \frac{\partial}{\partial x}(f f_x) + \frac{\partial}{\partial y}(f f_y) + \frac{\partial}{\partial z}(f f_z)$$
$$= f_x^2 + f_y^2 + f_z^2 + f\underbrace{(f_{xx} + f_{yy} + f_{zz})}_{= 0 \text{ on } E \ (\because f \text{ is harmonic})}$$
$$= \|F\|^2$$

$$\therefore \iint_{\partial E} fF \cdot n \, d\sigma = \iiint_E \|F\|^2 \, dV.$$

§13.5

Ex10 (a) $\nabla u = (u_{x_1}, u_{x_2}, \cdots, u_{x_m})$

$$\nabla \cdot \nabla u = \frac{\partial}{\partial x_1} u_{x_1} + \frac{\partial}{\partial x_2} u_{x_2} + \cdots + \frac{\partial}{\partial x_m} u_{x_m} = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_mx_m} = \Delta u.$$

(b). Apply Gauss's Thm with $F = u\nabla v$ to get

$$\iint_{\partial E} u\nabla v \cdot n \, d\sigma = \iint_E \operatorname{div}(u\nabla v)\, dV$$

$$= \iint_E \nabla \cdot (u\nabla v)\, dV \underset{\text{§13.5 Ex8.(c)}}{=} \iint_E \nabla u \cdot \nabla v + u(\nabla \cdot \nabla v)\, dV$$

$$\overset{(a)}{=} \iint_E (\nabla u \cdot \nabla v + u\Delta v)\, dV.$$

for all C^2 functions $u, v : E \to \mathbb{R}$.

(c). By (b),

$$\iiint_E (u\Delta v + \nabla u \cdot \nabla v)\, dV = \iint_{\partial E} u\nabla v \cdot n \, d\sigma$$

$$\iiint_E (v\Delta u + \nabla v \cdot \nabla u)\, dV = \iint_{\partial E} v\nabla u \cdot n \, d\sigma$$

$$\Rightarrow \iiint_E (u\Delta v - v\Delta u)\, dV = \iint_{\partial E} (u\nabla v - v\nabla u) \cdot n \, d\sigma.$$

for all C^2 functions $u, v : E \to \mathbb{R}$.

(d). From (b),

$$\iint_E (u\Delta u + \nabla u \cdot \nabla u)\, dV = \iint_{\partial E} u\nabla u \cdot n \, d\sigma = 0 \quad (\because u = 0 \text{ on } \partial E)$$

$$\parallel$$

$$\iint_E |\nabla u|^2 dV \quad (\because \Delta u = 0 \text{ in } E)$$

$|\nabla u|^2 \geq 0$ & $|\nabla u|^2$ is C^1 ($\because u$ is C^2)

$\Rightarrow |\nabla u|^2 \equiv 0$ in E $\Rightarrow \nabla u \equiv 0$ in E.

By mean value thm & $u = 0$ on ∂E, u is conti. on $\bar{E}$

$\Rightarrow u \equiv 0$ in $\bar{E}$.

§13.5

Ex10(e). ($\Rightarrow$). By Green's Thm,

$$\int_{\partial E} (u_x\,dy - u_y\,dx) = \iint_E \frac{\partial}{\partial x}u_x - \frac{\partial}{\partial y}(-u_y)\,dA = \iint_E \Delta u\,dA = 0$$

for all E satisfies the conditions.

($\Leftarrow$). Fix $(x_0, y_0) \in V$, $\quad V$ is open $\Rightarrow \exists\, r_0 > 0$ s.t. $B_r(x_0, y_0) \subset V \quad \forall 0<r<r_0$

$$\therefore\ 0 = \int_{\partial B_r(x_0, y_0)} (u_x\,dy - u_y\,dx) = \iint_{B_r(x_0,y_0)} \Delta u\,dA$$

$$\Rightarrow \frac{1}{\mathrm{Vol}(B_r(x_0,y_0))} \iint_{B_r(x_0,y_0)} \Delta u\,dA = 0$$

u is C^2 on $V \Rightarrow \Delta u$ is conti. at (x_0, y_0)

$$\overset{\text{§12.2 Ex5}}{\Longrightarrow}\quad \Delta u(x_0, y_0) = \lim_{r\to 0+} \frac{1}{\mathrm{Vol}(B_r(x_0,y_0))} \iint_{B_r(x_0,y_0)} \Delta u\,dA = 0.$$

(x_0, y_0) is arbitrary in $V \Rightarrow u$ is harmonic in V.

§ 13.6

Ex 3 (a) Gauss's Thm $\Rightarrow$

$$\iint_S F\cdot n\, d\sigma = \iiint_{B_1(0,0,0)} \operatorname{div} F\, dV$$

$$= \iiint_{B_1(0,0,0)} (z^2 + x^2 + y^2)\, dV$$

$$= \int_0^1 \int_0^{\pi} \int_0^{2\pi} r^2\, r^2 \sin\varphi\, d\theta\, d\varphi\, dr$$

$$= \left(\int_0^1 r^4\, dr\right)\left(\int_0^{\pi} \sin\varphi\, d\varphi\right)\left(\int_0^{2\pi} d\theta\right)$$

$$= \frac{1}{5}\cdot 2\cdot 2\pi = \frac{4}{5}\pi$$

(b) Stoke's Thm $\Rightarrow$

$$\iint_S F\cdot n\, d\sigma = \iint_S \operatorname{curl} G\cdot n\, d\sigma \qquad \text{where } G = \left(0, -xyz, -\frac{x^2 z}{2}\right)$$

$$= \int_{\partial S} G\cdot T\, ds = \int_0^{2\pi} -\frac{1}{2}\cos\theta\sin^2\theta\cdot\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{2\sqrt{2}}\cos^2\theta\sin\theta\cdot\frac{1}{\sqrt{2}}\cos\theta\, d\theta$$

$\Big(\partial S:\ y = z,\ x^2+y^2+z^2 = 1 \qquad x^2+2y^2 = 1.$

$\theta\in[0,2\pi].\ \left(\cos\theta, \frac{1}{\sqrt{2}}\sin\theta, \frac{1}{\sqrt{2}}\sin\theta\right) \qquad T = \left(-\sin\theta, \frac{1}{\sqrt{2}}\cos\theta, \frac{1}{\sqrt{2}}\cos\theta\right) \Big)$

$$= \frac{-\pi}{8\sqrt{2}}$$

(c) Let $E = \left\{(x,y,z)\in\mathbb{R}^3 \,\middle|\, 2\le y\le 4,\ x^2+z^2\le\frac{y^2}{4}\right\}$, a 3-dim J.R.

Then $\partial E = S \cup \left\{(x,y,z) \,\middle|\, y = 2 \vee 4,\ x^2+z^2\le\frac{y^2}{4}\right\}$

Gauss's Thm $\Rightarrow$

$$\iint_{\partial E} F\cdot n\, d\sigma = \iiint_E \operatorname{div} F\, dV = \iiint_E (1-2+1)\, dV = 0$$

$$\iint_S F\cdot n\, d\sigma + \iint_{B_2(0,0)} (x,-8,z)\cdot(0,1,0)\, d(x,z) - \iint_{B_1(0,0)} (x,-4,z)\cdot(0,1,0)\, d(x,z)$$

$$= \iint_S F\cdot n\, d\sigma - 8\iint_{B_2(0,0)} d(x,z) + 4\iint_{B_1(0,0)} d(x,z)$$

$$= \iint_S F\cdot n\, d\sigma - 8\cdot 4\pi + 4\cdot\pi = \iint_S F\cdot n\, d\sigma - 28\pi$$

$$\therefore \iint_S F\cdot n\, d\sigma = 28\pi.$$

§13.6

Ex3 (d). Let $E = \{(x,y,z) \mid x^2+y^2 \le 4,\ x^2+y^2-4 \le z \le 4-x^2-y^2\}$, a 3-dim J.R.

$\partial E = S$.

Gauss's Thm $\Rightarrow$

$$\iint_S F\cdot n\, d\sigma = \iiint_E \operatorname{div} F\, dV = \iiint_E 1+2y+1\, dV$$

$$= \iint_{B_2(0,0)} \int_{x^2+y^2-4}^{4-x^2-y^2} (2+2y)\, dz\, d(x,y)$$

$$= \iint_{B_2(0,0)} 4(4-x^2-y^2)(1+y)\, d(x,y)$$

$$= \int_0^2 \int_0^{2\pi} 4(4-r^2)(1+r\sin\theta)\, r\, d\theta\, dr$$

$$= \int_0^2 4(4-r^2)\, r\cdot 2\pi\cdot dr \qquad \left(\int_0^{2\pi}(1+r\sin\theta)\, d\theta = 2\pi\right)$$

$$= 32\pi.$$

(e). Let $E = \{(x,y,z) \mid x^2+y^2 \le z,\ 0 \le z \le 2 \ \vee\ x^2+y^2 \le 2,\ 2 \le z \le 5$

$\vee\ x^2+y^2 \le 7-z,\ 5 \le z \le 6\}$

Then $\partial E = S \cup \{(x,y,z) \mid z=6,\ x^2+y^2 \le 1\}$

Gauss's Thm $\Rightarrow$

$$\iint_{\partial E} F\cdot n\, d\sigma = \iiint_E \operatorname{div} F\, dV = \iiint_E (0+0+0)\, dV = 0$$

$$\parallel$$

$$\iint_S F\cdot n\, d\sigma + \iint_{B_1(0,0)} (2y,\ 12,\ 1)\cdot(0,0,1)\, d(x,y)$$

$$= \iint_S F\cdot n\, d\sigma + \iint_{B_1(0,0)} d(x,y)$$

$$= \iint_S F\cdot n\, d\sigma + \pi$$

$$\Rightarrow \iint_S F\cdot n\, d\sigma = -\pi.$$

§13.6

Ex7. (a). By Stoke's Thm,

$$0=\iint_S \operatorname{curl} F \cdot n \, d\sigma = \int_{\partial S} F \cdot T \, ds.$$

The angle between $T(x_0)$ & $F(x_0)$ is never obtuse for any $x_0 \in \partial S$

$\Rightarrow F(x_0) \cdot T(x_0) \geq 0$ for any $x_0 \in \partial S$

F is C^1, T is smooth, $\int_{\partial S} F \cdot T \, ds = 0$

$\Rightarrow F \cdot T \equiv 0$ on ∂S.

i.e. $T(x_0)$ & $F(x_0)$ are orthogonal $\forall x_0 \in \partial S$.

(b). $F_R \to F$ uniformly on ∂S

$$\Rightarrow \lim_{R\to\infty} \int_{\partial S} F_R \cdot T \, ds = \int_{\partial S} F \cdot T \, ds \qquad (\S 13.1 \text{ Ex } 7(a))$$

|| Stoke's Thm || Stoke's Thm

$$\lim_{R\to\infty} \iint_S \operatorname{Curl} F_R \cdot n \, d\sigma \qquad \iint_S \operatorname{Curl} F \cdot n \, d\sigma$$

§13.6

Ex 8. (a) ⇒ (b):

$F=(P,Q)=\nabla f$ for some $f: E \to \mathbb{R}$.

F is $C^1 \Rightarrow f$ is C^2.

$\therefore f_{xy} = f_{yx}$ in E

$$\begin{array}{cc} f_{xy} & f_{yx} \\ \| & \| \\ \frac{\partial}{\partial y} f_x & \frac{\partial}{\partial x} f_y \\ \| & \| \\ \frac{\partial}{\partial y} P & \frac{\partial}{\partial x} Q \end{array}$$

(b) ⇒ (c)

Let C, Ω as in the statements in (c).

$$\int_C F\cdot T\,ds = \iint_\Omega (Q_x - P_y)\,dA = \iint_\Omega 0\,dA = 0 \quad \text{by Green's Thm.}$$

(c) ⇒ (a)

For each $(x,y) \in \mathring{E}$,

define $f(x,y) = \int_{C_1(x,y)} F\cdot T\,ds + \int_{C_2(x,y)} F\cdot T\,ds$

where $C_1(x,y)$ is the segment from $(0,0)$ to $(x,0)$
$C_2(x,y)$ is the segment from $(x,0)$ to (x,y).

$$f(x,y) = \int_0^x P(u,0)\,du + \int_0^y Q(x,v)\,dv$$

$f_y(x,y) = Q(x,y)$ in $\mathring{E}$.

$\exists\, r>0$ s.t. $\overline{B_r(x,y)} \in \mathring{E} \Rightarrow (x+h, y) \in E \quad \forall\, |h| \le r$

$\therefore [x-r, x+r] \times [0, y] \subset E$ if $y > 0$

$[x-r, x+r] \times [y, 0] \subset E$ if $y < 0$

WLOG, suppose $y \ge 0$, let Ω_h be the rectangle whose vertices are $(x,0)$, (x,y), $(x+h, 0)$, $(x+h, y)$ for $|h| \le r$. $\Omega_h \subset E$.

By (c), $\int_{\partial\Omega_h} F\cdot T\,ds = 0$

$$\Rightarrow \int_x^{x+h} P(u,0)\,du + \int_0^y Q(x+h, v)\,dv - \int_x^{x+h} P(u,y)\,du - \int_0^y Q(x,v)\,dv =$$

$\therefore \dfrac{f(x+h,y)-f(x,y)}{h} = \dfrac{1}{h}\left\{ \int_x^{x+h} P(u,0)\,du + \int_0^y Q(x+h,v)\,dv - \int_0^y Q(x,v)\,dv \right\}$

$= \dfrac{1}{h}\int_x^{x+h} P(u,y)\,du$

$\longrightarrow P(x,y)$ as $h \to 0$.

$\therefore f_x(x,y) = P(x,y)$.

If $y=0$,

$f(x,0) = \int_0^x P(u,0)\,du$

$f_x(x,0) = P(x,0)$.

$\therefore F = \nabla f$ in $\mathring{E}$.

§13.6

Ex 9. (a) ⇒ (b).

$F = \operatorname{curl} G = (R_y - Q_z, P_z - R_x, Q_x - P_y)$ for $G = (P, Q, R)$ on Ω.

$$\operatorname{div} F = \frac{\partial}{\partial x}(R_y - Q_z) + \frac{\partial}{\partial y}(P_z - R_x) + \frac{\partial}{\partial z}(Q_x - P_y)$$
$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0 \quad \text{since } G \text{ is } C^2.$$

$$\therefore \iint_S F \cdot n \, d\sigma = \iiint_E \operatorname{div} F \, dV = \iiint_E 0 \, dV = 0 \quad \text{by Gauss's Thm.}$$

(b) ⇒ (c).

Fix $(x_0, y_0, z_0) \in \mathring{\Omega}$, $\exists r_0 > 0$ s.t. $B_r(x_0, y_0, z_0) \subset \Omega$. $\forall r \le r_0$

Take $E = B_r(x_0, y_0, z_0)$ in (b),

$$0 = \iint_{\partial B_r(x_0, y_0, z_0)} F \cdot n \, d\sigma = \iiint_{B_r(x_0, y_0, z_0)} \operatorname{div} F \, dV$$

$$\Rightarrow \quad 0 = \frac{1}{\operatorname{Vol}(B_r(x_0, y_0, z_0))} \iiint_{B_r(x_0, y_0, z_0)} \operatorname{div} F \, dV$$

$$\to \operatorname{div} F(x_0, y_0, z_0) \quad \text{as } r \to 0 \quad \text{by §12.2 Ex 5.}$$

$\therefore \operatorname{div} F \equiv 0$ in $\mathring{\Omega} \Rightarrow \operatorname{div} F \equiv 0$ in Ω by continuity of DF.

(c) ⇒ (a).

Define $G: \Omega \to \mathbb{R}^3$, $G = (P, Q, R)$ by

$$P(x, y, z) = \int_0^z F_2(x, y, w)\, dw - \int_0^y F_3(x, v, 0)\, dv$$

$$Q(x, y, z) = -\int_0^z F_1(x, y, w)\, dw$$

$$R(x, y, z) \equiv 0.$$

Then
$$R_y - Q_z = -Q_z = F_1$$
$$P_z - R_x = P_z = F_2$$
$$Q_x - P_y = -\int_0^z F_{1x}(x, y, w)\, dw - \int_0^z F_{2y}(x, y, w)\, dw + F_3(x, y, 0)$$
$$= \int_0^z F_{3z}(x, y, w)\, dw + F_3(x, y, 0) \qquad (\because \operatorname{div} F = 0)$$
$$= F_3(x, y, z).$$

i.e. $F = \operatorname{curl} G$.

§13.6

Ex 10. (a) Stoke's Thm $\Rightarrow$

$$\iint_{\partial S} (fF)\cdot T\, ds = \iint_S \operatorname{curl}(fF)\cdot n\, d\sigma$$

$$= \iint_S \nabla\times(fF)\cdot n\, d\sigma \underset{\substack{\uparrow \\ \text{§13.5 Ex 8(b).}}}{=} \iint_S (f(\nabla\times F) + \nabla f\times F)\cdot n\, d\sigma$$

$$= \iint (f \underbrace{\operatorname{curl} F}_{\substack{\shortparallel \\ 0}} + \nabla f\times\nabla f)\cdot n\, d\sigma = 0 \qquad (\vec{a}\times\vec{a} = 0 \quad \forall \vec{a}\in\mathbb{R}^3)$$

by §13.5 Ex 9(a)

(b). Gauss' Thm $\Rightarrow$

$$\iint_{\partial E} (fF)\cdot n\, d\sigma = \iiint_E \operatorname{div}(fF)\, dV$$

$$= \iiint_E \nabla\cdot(fF)\, dV \underset{\substack{\uparrow \\ \text{§13.5 Ex 8(c).}}}{=} \iiint_E (\nabla f\cdot F + f\cdot(\nabla\cdot F))\, dV$$

Thm 13.6.1 $\Rightarrow$ $\nabla\cdot F = \operatorname{div} F = 0$ since $F = \operatorname{curl} G$ on E.

$$\therefore \iint_{\partial E} (fF)\cdot n\, d\sigma = \iiint_E \nabla f\cdot F\, dV$$

§13.6

Ex11(a). $F=(P,Q)$ F is exact on $\mathbb{R}^2\setminus\{(0,0)\}$

i.e. $Q_x = P_y$ on $\mathbb{R}^2\setminus\{(0,0)\}$

WLOG, may assume that C_1 is the outer boundary, C_2 is the inner boundary of E.

Green's Thm $\Rightarrow$

$$\int_{\partial E} F\cdot T ds = \iint_E (Q_x - P_y)\, dA = 0$$

$$= \int_{C_1} F\cdot T ds - \int_{C_2} F\cdot T ds$$

$$\Rightarrow \int_{C_1} F\cdot T ds = \int_{C_2} F\cdot T ds.$$

(b). For any $r>0$ s.t. $B_r(0,0)\subset E^\circ$, consider $E_r = E\setminus B_r(0,0)$,

$$Q_x = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$P_y = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} = Q_x$$

$\therefore F$ is exact on $\mathbb{R}^2\setminus\{0,0)\}$.

By (a) $\displaystyle\int_{\partial E} F\cdot T ds = \int_{\partial B_r(0,0)} F\cdot T ds$

$$= \int_0^{2\pi} \left(\frac{-r\sin\theta}{r^2}, \frac{r\cos\theta}{r^2}\right)\cdot(-r\sin\theta, r\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} d\theta = 2\pi.$$

(c). Suppose that E_1, E_2 are two simple-connected 3-dim regions s.t. $E_1\subset \mathring{E}_2$, and $\partial E_1 \equiv S_1$, $\partial E_2 \equiv S_2$ are piecewise smooth C' surfaces oriented positively. If $F:\mathbb{R}^3\setminus\{(0,0,0)\}\to\mathbb{R}$ is a C' function s.t. $\operatorname{div}F=0$ (i.e. $F=\operatorname{curl} G$ for some G (by Thm 13.61)), then

$$\iint_{S_1} F\cdot n\, d\sigma = \iint_{S_2} F\cdot n\, d\sigma$$

provided that $(0,0,0)\notin E_2\setminus E_1$.