§12.1

#2 (a) Suppose $\#E < \infty$, $E = \{x_1, x_2, \cdots, x_N\} = \overline{E}$.

Given $\varepsilon > 0$, let Q_k be the cube with center x_k and side length $\left(\frac{\varepsilon}{2N}\right)^{\frac{1}{n}}$, then

$$\overline{E} = \bigcup_{k=1}^{N} \{x_k\} \subset \bigcup_{k=1}^{N} Q_k,$$

$$\sum_{k=1}^{N} |Q_k| = \sum_{k=1}^{N} \left[\left(\frac{\varepsilon}{2N}\right)^{\frac{1}{n}}\right]^n = \frac{\varepsilon}{2} < \varepsilon.$$

By Thm 12.8, E is a Jordan region of volume zero.

(b). Let $E = \{(x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Q} \cap [0, 1]\}$, then E is countable.

$\partial E = [0,1] \times [0,1]$ $\therefore$ $Vol(\partial E) = 1$

$\Rightarrow E$ is not a Jordan region.

(c). Let $E = \{(x, c) \mid a \le x \le b\}$

Given $\varepsilon > 0$, let $R = [a, b] \times \left[c - \frac{\varepsilon}{2(b-a)+1}, c + \frac{\varepsilon}{2(b-a)+1}\right]$, a grid $\mathcal{G} = \{R\}$. Then $R \supset E$ and

$$V(\partial E; \mathcal{G}) = V(E; \mathcal{G})$$
$$= |R| = (b-a) \times \frac{2\varepsilon}{2(b-a)+1} < \varepsilon$$

$\therefore$ By Definition 12.3, E is a Jordan region.

The case $\{(c, y) \mid a \le y \le b\}$ is similar.

§12.1

#4. E_1, E_2 are Jordan regions in $\mathbb{R}^n$.

(a). For any rectangle $R \supset E_2 \supset E_1$ with a grid $\mathcal{G}$, Rmk 12.1(ii) $\Rightarrow$

$$V(E_1; \mathcal{G}) \leq V(E_2; \mathcal{G})$$

$$\Rightarrow \mathrm{Vol}(E_1) \leq \mathrm{Vol}(E_2)$$

(b). E_1, E_2 are Jordan regions $\Leftrightarrow \mathrm{Vol}(\partial E_1) = \mathrm{Vol}(\partial E_2) = 0$

$\partial(E_1 \cap E_2) \subseteq \partial E_1 \cup \partial E_2$, and

$\partial(E_1 \setminus E_2) = \partial(E_1 \cap E_2^c) \subseteq \partial E_1 \cup \partial(E_2^c) = \partial E_1 \cup \partial E_2$

$$\Rightarrow 0 \leq \begin{matrix} \mathrm{Vol}(\partial(E_1 \cap E_2)) \\ \mathrm{Vol}(\partial(E_1 \setminus E_2)) \end{matrix} \leq \mathrm{Vol}(\partial E_1 \cup \partial E_2) \leq \mathrm{Vol}(\partial E_1) + \mathrm{Vol}(\partial E_2) = 0$$

(first $\leq$ by (a); second $\leq$ by Corollary 12.9)

$$\Rightarrow \mathrm{Vol}(\partial(E_1 \cap E_2)) = \mathrm{Vol}(\partial(E_1 \setminus E_2)) = 0$$

$\Leftrightarrow E_1 \cap E_2$ and $E_1 \setminus E_2$ are Jordan regions.

(c). By Corollary 12.9, $\mathrm{Vol}(E_1 \cup E_2) \leq \mathrm{Vol}(E_1) + \mathrm{Vol}(E_2)$.

Hence it's sufficient to show that $\mathrm{Vol}(E_1 \cup E_2) \geq \mathrm{Vol}(E_1) + \mathrm{Vol}(E_2)$.

For arbitrary rectangles R_1, R_2 with grids $\mathcal{G}_1 = \{R_1^{(1)}, R_2^{(1)}, \cdots, R_{\ell_1}^{(1)}\}$, $\mathcal{G}_2 = \{R_1^{(2)}, R_2^{(2)}, \cdots, R_{\ell_2}^{(2)}\}$ and $R_1 \supset E_1 \setminus E_2$, $R_2 \supset E_2$.

Let R be a rectangle containing R_1 & R_2 with a grid $\mathcal{G} = \{R_1^{(0)}, R_2^{(0)}, \cdots, R_{\ell_0}^{(0)}\}$ s.t. $\mathcal{G}$ is finer than $\mathcal{G}_1$ and $\mathcal{G}_2$ if $\mathcal{G}$ is restricted on R_1 and R_2 respectively.

Then
$$v(E_1 \setminus E_2; \mathcal{G}_1) + v(E_2; \mathcal{G}_2)$$
$$\leq v(E_1 \setminus E_2; \mathcal{G} \text{ restricted on } R_1) + v(E_2; \mathcal{G} \text{ restricted on } R_2)$$
$$= \sum_{R_j^{(0)} \subset (E_1 \setminus E_2)^\circ} |R_j^{(0)}| + \sum_{R_j^{(0)} \subset E_2^\circ} |R_j^{(0)}|$$
$$\leq \sum_{R_j^{(0)} \subset (E_1 \cup E_2)^\circ} |R_j^{(0)}|$$

$\left(\because E_1 \cup E_2 = (E_1 \setminus E_2) \cup E_2,\ (E_1 \setminus E_2) \cap E_2 = \phi \Rightarrow (E_1 \cup E_2)^\circ \supseteq (E_1 \setminus E_2)^\circ \cup E_2^\circ \text{ and } (E_1 \setminus E_2)^\circ \cap E_2^\circ = \phi \right)$

$$= v(E_1 \cup E_2; \mathcal{G})$$
$$\leq \mathrm{Vol}(E_1 \cup E_2)$$

$$\Rightarrow \mathrm{Vol}(E_1 \cup E_2) \geq \mathrm{Vol}(E_1 \setminus E_2) + \mathrm{Vol}(E_2) = \mathrm{Vol}(E_1 \setminus E_2) + \mathrm{Vol}(E_1 \cap E_2) + \mathrm{Vol}(E_2$$

$$\geq \mathrm{Vol}((E_1 \setminus E_2) \cup (E_1 \cap E_2)) + \mathrm{Vol}(E_2) = \mathrm{Vol}(E_1) + \mathrm{Vol}(E_2)$$

§12.1

#4 (d). $E_2 \subseteq E_1 \Rightarrow E_1 = E_2 \cup (E_1 \setminus E_2)$ and $E_2 \cap (E_1 \setminus E_2) = \phi$

$\overset{(c)}{\Rightarrow} Vol(E_1) = Vol(E_2) + Vol(E_1 \setminus E_2)$

$\Rightarrow Vol(E_1 \setminus E_2) = Vol(E_1) - Vol(E_2)$.

(e). $E_1 \cup E_2 = (E_1 \setminus E_2) \cup E_2$ and $(E_1 \setminus E_2) \cap E_2 = \phi$

$\overset{(c)}{\Rightarrow} Vol(E_1 \cup E_2) = Vol(E_1 \setminus E_2) + Vol(E_2)$

$= Vol(E_1) - Vol(E_1 \cap E_2) + Vol(E_2)$

↑ by (d) $\because E_1 \supset (E_1 \cap E_2)$ and $E_1 \setminus (E_1 \cap E_2) = E_1 \setminus E_2$

#5 (a). E is a Jordan region in $\mathbb{R}^n$ (by Rmk 12.6. (ii))

$\Leftrightarrow Vol(\partial E) = 0$

$\quad\quad \parallel$

$Vol(\partial(E^\circ)) = Vol(\partial \bar{E})$

$\Leftrightarrow E^\circ$ and $\bar{E}$ are Jordan regions.

(b). $\bar{E} = E^\circ \cup \partial E$ and $E^\circ \cap \partial E = \phi$, then by (a) and #4(a)(c),

$Vol(\bar{E}) = Vol(E^\circ) + Vol(\partial E) = Vol(E^\circ)$

$\quad \vee\!/$

$Vol(E) \geq Vol(E^\circ)$

$\Rightarrow Vol(E^\circ) = Vol(E) = Vol(\bar{E})$

(c). $(\Rightarrow)$ $0 < Vol(E) = \sup \{ v(E; \mathcal{G}) \mid \mathcal{G}$ ranges over all grids on $R\}$

$\Rightarrow \exists \mathcal{G}$ s.t. $v(E; \mathcal{G}) = \sum_{R_j \subset E^\circ} |R_j| > 0$

$\Rightarrow \exists R_j \subset E^\circ \Rightarrow E^\circ \neq \phi$.

$(\Leftarrow)$ $E^\circ \neq \phi \Rightarrow \exists x \in E^\circ$

E° is open $\Rightarrow \exists r > 0$ s.t. $B_r(x) \subset E^\circ$

$\Rightarrow$ the cube Q with center x, side length $\frac{r}{2\sqrt{n}}$ is contained in $B_r(x)$ hence in E°.

$\Rightarrow Vol(E^\circ) \geq Vol(Q) = \left(\frac{r}{2\sqrt{n}}\right)^n > 0$.

↗ #4(a)

§12.1

#5(d)(e) Let $G=\{(x, f(x)) \mid x\in[a,b]\}$ be the graph of $y=f(x)$, $x\in[a,b]$

$f:[a,b]\to\mathbb{R}$ is integrable on $[a,b]$

$\Rightarrow$ Given $\varepsilon>0$, $\exists$ a Partition $P=\{a=x_0<x_1<\cdots<x_m=b\}$ of $[a,b]$

s.t. $U(f,P)-L(f,P)<\varepsilon$.

Since $G\subset \bigcup_{k=1}^{m}[x_{k-1},x_k]\times[m_k,M_k]$, Rmk 12.1(ii), Def 12.13

& Corollary 12.9 $\Rightarrow$

$$\overline{Vol}(G)\le\sum_{k=1}^{m}Vol([x_{k-1},x_k]\times[m_k,M_k])$$

$$=\sum_{k=1}^{m}(x_k-x_{k-1})(M_k-m_k)$$

$$=U(f,P)-L(f,P)<\varepsilon$$

$\Rightarrow \overline{Vol}(G)=0 \overset{\text{Thm 12.14}}{\Longrightarrow}$ G is a Jordan region and $Vol(G)=0$.

$\quad\;\; \vee|$

$\underline{Vol}(G)\ge 0$

<u>Case</u> f is conti. on $[a,b]$ $\Rightarrow$ f is integrable on $[a,b]$

$\Rightarrow$ the graph of $y=f(x)$, $x\in[a,b]$ is a Jordan region in $\mathbb{R}^n$ and of volume zero.

If we just assume that f is bounded, the graph of $y=f(x)$, $x\in[a,b]$ may be not a Jordan region.

For example, let $[a,b]=[0,1]$, define $f:[0,1]\to\mathbb{R}$ by

$$f(x)=\begin{cases} p\sqrt{2} \pmod 1 & \text{if } x=\frac{q}{p}\in\mathbb{Q},\ p,q\in\mathbb{N},\ (p,q)=1 \\ 0 & \text{otherwise.}\end{cases}$$

check by yourself: Then f is bdd, and the graph G of $y=f(x)$, $x\in[a,b]$ is dense in $[0,1]\times[0,1]$ with $G^\circ=\phi$, hence $\partial G=[0,1]\times[0,1]$, $Vol(\partial G)=1\ne 0 \Rightarrow G$ is not a Jordan region.

§12.1

#8. Case: E has no cluster points. $\qquad$ $m \in \mathbb{N}$

By P. 263 §9.1 Ex #10 (b), E is finite, say $E = \{x_1, x_2, \cdots, x_m\}$.

For $\varepsilon > 0$, let Q_k be the cube with center x_k, side length $\left(\frac{\varepsilon}{2m}\right)^{\frac{1}{n}}$.

Then $\overline{E} = E \subset \bigcup_{k=1}^{m} Q_k$ and $\sum_{k=1}^{m} |Q_k| = \sum_{k=1}^{m} \frac{\varepsilon}{2m} = \frac{\varepsilon}{2} < \varepsilon$.

Thm 12.8 $\Rightarrow$ E is a Jordan region of volume zero.

Case: E has cluster points $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_m$.

Obviously, $\overline{E} \subset E \cup \{\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_m\}$.

Given $\varepsilon > 0$, for each $1 \le k \le m$, let $\widetilde{Q}_k$ be the cube with center $\overline{x}_k$ and side length $\left(\frac{\varepsilon}{4m}\right)^{\frac{1}{n}}$. Consider $\widetilde{E} = E \setminus \bigcup_{k=1}^{m} \widetilde{Q}_k$.

Claim: $\widetilde{E}$ is finite.

pf: Suppose the contrary, i.e. $\widetilde{E}$ is infinite.

$\widetilde{E} \subset E \Rightarrow \widetilde{E}$ is bounded.

P. 263 §9.1 Ex #10 (b) $\Rightarrow$ $\widetilde{E}$ has at least one cluster point.

But $\widetilde{E} \subset E \Rightarrow$ A cluster pt of $\widetilde{E}$ must be a cluster pt of E.

Hence $\widetilde{E}$ has a cluster pt in $\{\overline{x}_1, \cdots, \overline{x}_m\}$, say $\overline{x}_{k_0}$.

But $\widetilde{E} \cap \widetilde{Q}_{k_0} = \phi \Rightarrow \overline{x}_{k_0}$ is not a cluster pt of $\widetilde{E}$ $\divideontimes$.

Therefore, $\widetilde{E}$ is finite.

If $\widetilde{E} = \phi$, then $\overline{E} \subset \{\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_m\} \subset \bigcup_{k=1}^{m} \widetilde{Q}_k$, and $\sum_{k=1}^{m} |\widetilde{Q}_k| = \frac{\varepsilon}{2} < \varepsilon$.

Thm 12.8 $\Rightarrow$ E is a Jordan region of volume zero.

$\ell \in \mathbb{N}$ If $\widetilde{E} = \{x_{m+1}, x_{m+2}, \cdots, x_{m+\ell}\}$. For $m+1 \le k \le m+\ell$, let Q_k be the cube with center x_k, side length $\frac{1}{\ell}\left(\frac{\varepsilon}{4m}\right)^{\frac{1}{n}}$. For $1 \le k \le m$, split each $\widetilde{Q}_k$ into ℓ^n equal cubes, denoted by $Q_1^{(k)}, Q_2^{(k)}, \cdots, Q_{\ell^n}^{(k)}$.

(Note that each $Q_j^{(k)}$ has side length $\frac{1}{\ell}\left(\frac{\varepsilon}{4m}\right)^{\frac{1}{n}}$)

Then $\overline{E} \subset \widetilde{E} \cup \left(\bigcup_{k=1}^{m} \widetilde{Q}_k\right) \subset \left(\bigcup_{k=m+1}^{m+\ell} Q_k\right) \cup \left(\bigcup_{k=1}^{m} \left(\bigcup_{j=1}^{\ell^n} Q_j^{(k)}\right)\right)$

and $\sum_{k=m+1}^{m+\ell} |Q_k| + \sum_{k=1}^{m} \sum_{j=1}^{\ell^n} |Q_j^{(k)}| = (\ell + m\ell^n) \frac{1}{\ell^n} \frac{\varepsilon}{4m} = \frac{\varepsilon}{4m\ell^{n-1}} + \frac{\varepsilon}{4} \le \frac{\varepsilon}{2} < \varepsilon$.

Thm 12.8 $\Rightarrow$ E is a Jordan region of volume zero.

∘

§12.1

#9(a). Obvious by the definitions. (Check by yourself).

(b). Let $E = \{x_k\}_{k=1}^{\infty}$.

For each $\varepsilon > 0$, let R_k be the cube with center x_k, side length $\left(\frac{\varepsilon}{2^{k+1}}\right)^{\frac{1}{n}}$.

Then $E = \{x_k\}_{k=1}^{\infty} \subset \bigcup_{k=1}^{\infty} R_k$ and $\sum_{k=1}^{\infty} |R_k| = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{2} < \varepsilon$.

i.e. E is of measure zero.

(c). Let $E = \{(x,y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Q} \cap [0,1]\}$. $\partial E = [0,1] \times [0,1]$.

E is countable $\Rightarrow$ E is of measure zero by (b).

But $Vol(\partial E) = Vol([0,1] \times [0,1]) = 1 \neq 0 \Rightarrow E$ is not a Jordan region.

03.12.2

#5

f is conti. at x_0 and E is open

Given $\varepsilon>0$, $\exists\,\delta>0$ s.t. $|f(x)-f(x_0)|<\frac{\varepsilon}{2}$ if $\|x-x_0\|<\delta$ and $B_\delta(x_0)\subset E$.

Then for all $0<r<\delta$,

$$\left|\frac{1}{Vol(B_r(x_0))}\int_{B_r(x_0)} f(x)\,dx - f(x_0)\right|$$

$$\le \frac{1}{Vol(B_r(x_0))}\int_{B_r(x_0)} |f(x)-f(x_0)|\,dx \qquad (\text{Thm } 12.25\,(iii))$$

$$\le \frac{1}{Vol(B_r(x_0))}\cdot\frac{\varepsilon}{2}\cdot Vol(B_r(x_0)) = \frac{\varepsilon}{2}<\varepsilon \qquad (\text{Thm } 12.25\,(ii))$$

$$\therefore \lim_{r\to 0+}\frac{1}{Vol(B_r(x_0))}\int_{B_r(x_0)} f(x)\,dx = f(x_0).$$

(Note that $B_r(x_0)$ is a Jordan domain for each $r>0$, and f is integrable on $B_r(x_0)$ if $0<r<\delta$ by Thm 12.29 (i))

#6(a)

① f is bdd.

Since $f_k\to f$ uniformly on E as $k\to\infty$.

For $1>0$, $\exists\,K\in\mathbb{N}$ s.t. $|f(x)-f_k(x)|<\varepsilon$ $\forall\,x\in E$, $k\ge K$.

$\therefore |f(x)|\le |f_K(x)|+1\le \sup_{x\in E}|f_K(x)|+1<\infty$ for $x\in E$ since f_K is bounded on E ($\because$ f_K is integrable on E)

② f is conti. at the points at which f_k is conti. for each $k\in\mathbb{N}$

Suppose f_k is conti. at some $x_0\in E$ for each $k\in\mathbb{N}$.

Given $\varepsilon>0$, $\exists\,K\in\mathbb{N}$ s.t. $|f(x)-f_k(x)|<\frac{\varepsilon}{3}$ $\forall x\in E$, $k\ge K$.

$\therefore |f(x)-f_K(x)|<\frac{\varepsilon}{3}$ $\forall\,x\in E$.

f_K is conti. at x_0

$\Rightarrow \exists\,\delta>0$ s.t. $|f_K(x)-f_K(x_0)|<\frac{\varepsilon}{3}$ if $\|x-x_0\|<\delta$ and $x\in E$.

$\therefore |f(x)-f(x_0)|\le|f(x)-f_K(x)|+|f_K(x)-f_K(x_0)|+|f_K(x_0)-f(x_0)|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$

if $\|x-x_0\|<\delta$ and $x\in E$.

i.e. f is conti. at x_0

③ the set of pts of discontinuity of f on E is of measure zero.

Let $Z = \{x \in E \mid f \text{ is not conti. at } x\}$.

$Z_k = \{x \in E \mid f_k \text{ is not conti. at } x\}$.

f_k is integrable on $E \Rightarrow Z_k$ is of measure zero by Thm 12.29 (i).

By ②, $Z^c \supset \bigcap_{k=1}^{\infty} Z_k^c \Rightarrow Z \subset \bigcup_{k=1}^{\infty} Z_k$

By Rmk 12.28, $\bigcup_{k=1}^{\infty} Z_k$ is of measure zero. Then, it's obvious that Z is of measure zero just by the definition.

①+③ and Thm 12.29 (i) $\Rightarrow f$ is integrable on E.

Given $\varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $|f(x) - f_k(x)| < \dfrac{\varepsilon}{Vol(E)+1}$ $\forall x \in E$ $k \geq K$.

$\therefore$ For $k \geq K$,

$$\left| \int_E f_k(x)\,dx - \int_E f(x)\,dx \right| \leq \int_E |f_k(x) - f(x)|\,dx \quad (\text{Thm } 12.25 (iii))$$

$$\leq \frac{\varepsilon}{Vol(E)+1} Vol(E) < \varepsilon \quad (\text{Thm } 12.25 (ii))$$

i.e. $\lim_{k \to \infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx$.

(b). Define $f_k : E \to \mathbb{R}$ by $f_k(x,y) = \cos\left(\frac{x}{k}\right) e^{\frac{y}{k}}$ for $(x,y) \in E$.

E is bdd $\Rightarrow \exists M > 0$ s.t. $|x| \leq M$, $|y| \leq M$ if $(x,y) \in E$.

$\cos x\, e^y$ is conti. at $(0,0) \Rightarrow$ for each $\varepsilon > 0$, $\exists \delta > 0$ s.t.

$|\cos x\, e^y - 1| = |\cos x\, e^y - \cos 0\, e^0| < \varepsilon$ if $(x,y) \in B_\delta(0,0)$.

Choose $K \in \mathbb{N}$ s.t. $\dfrac{M}{K} < \dfrac{\delta}{\sqrt{2}}$, then $\forall (x,y) \in E$, $|x| < M, |y| < M$

$\Rightarrow \sqrt{\left(\frac{x}{k}\right)^2 + \left(\frac{y}{k}\right)^2} \leq \frac{M}{k}\sqrt{2} \leq \frac{M}{K}\sqrt{2} < \delta$ if $k \geq K$.

i.e. $\left(\frac{x}{k}, \frac{y}{k}\right) \in B_\delta(0,0)$ if $k \geq K$

$\Rightarrow |f_k(x,y) - 1| = \left|\cos\frac{x}{k} e^{\frac{y}{k}} - 1\right| < \varepsilon$ $\forall (x,y) \in E$ if $k \geq K$

i.e. $f_k \to 1$ uniformly on E as $k \to \infty$

By (a), $\lim_{k \to \infty} \iint_E \cos\left(\frac{x}{k}\right) e^{\frac{y}{k}}\,dA = \iint_E dA = Vol(E)$.

#10. Suppose that $\exists\, x_0 \in V$ s.t. $f(x_0) > 0$. f is conti. at x_0 and V is open

$\Rightarrow \exists\, \delta > 0$ s.t. $f(x) > \frac{f(x_0)}{2} > 0$ if $\|x - x_0\| < \delta$, and $B_\delta(x_0) \subset V$.

Obviously, $B_\delta(x_0)$ is a Jordan region and $B_\delta(x_0) \subset V$, hence

$$0 = \int_{B_\delta(x_0)} f(x)\,dx \geq \frac{f(x_0)}{2} \cdot Vol(B_\delta(x_0)) > 0 \quad \times$$

$\therefore f(x) \leq 0 \quad \forall x \in V$.

Similarly, $f(x) \geq 0 \quad \forall x \in V$

$\Rightarrow f \equiv 0$ on V.

#11. E is a Jordan domain and $f: E \to \mathbb{R}$ is integrable.

(a) $f(E) \subseteq H$ for some cpt set H. $\phi: H \longrightarrow \mathbb{R}$ is conti.

ϕ is conti. on the cpt set $H \Rightarrow \phi$ is bdd. $\Rightarrow \phi \circ f$ is bdd. on E.

Let $Z = \{x \in E \mid f$ is not conti. at $x\}$.

f is integrable on $E \Rightarrow Z$ is of measure zero. (Thm 12.29 (i))

But for each $x \in E \setminus Z$, f is conti. at x. ϕ is conti. at $f(x)$

$\Rightarrow \phi \circ f$ is conti. at $x \Rightarrow$ The set of pts of discontinuity of $\phi \circ f$ is contained in Z, hence is of measure zero.

$\Rightarrow \phi \circ f$ is integrable on E.

(b). Let $E = [0,1] \times [0,1]$, $f(x, y) = x$ for $(x, y) \in E$.

f is conti. on $E \Rightarrow f$ is integrable on E.

$f(E) = [0,1] := H$. H is compact.

Define $\phi: H \to \mathbb{R}$ by $\phi(t) = \begin{cases} \frac{1}{t} & \text{if } t \in (0,1] \\ 0 & \text{if } t = 0 \end{cases}$.

Then ϕ is conti. on H except 0.

But $\phi \circ f$ is unbounded on E hence is not integrable.

§12.3

#5. (a) It's easy to show that $f(x,\cdot)$ is conti. on $[c,d]$ for each $x\in[a,b]$, hence is integrable on $[c,d]$ for each $x\in[a,b]$; $f(\cdot,y)$ is conti. on $[a,b]$ for each $y\in[c,d]$ hence is integrable on $[a,b]$ for each $y\in[c,d]$.

f is conti. on $R \Rightarrow f$ is integrable on R.

$\therefore$ The hypotheses of Fubini's Thm hold.

(b). Define $f:[0,1]\times[0,1]\longrightarrow\mathbb{R}$ by

$$f(x,y)=\begin{cases}2^{2n}\sin(2^n\pi x)\sin(2^n\pi y) & \text{for } (x,y)\in[2^{-n},2^{-n+1}]\times[2^{-n},2^{-n+1}],\ n\in\mathbb{N}\\ -2^{2n+1}\sin(2^{n+1}\pi x)\sin(2^n\pi y) & \text{for } (x,y)\in[2^{-n-1},2^{-n}]\times[2^{-n},2^{-n+1}],\ n\in\mathbb{N}\\ 0 & \text{otherwise.}\end{cases}$$

Obviously, f is conti. in each variable separately.

For each $0<y\leq 1$, $\exists\, n\in\mathbb{N}$ s.t. $y\in[2^{-n},2^{-n+1}]$,

$$\int_0^1 f(x,y)\,dx$$
$$=\sin(2^n\pi y)\left[-2^{2n+1}\int_{2^{-n-1}}^{2^{-n}}\sin(2^{n+1}\pi x)\,dx+2^{2n}\int_{2^{-n}}^{2^{-n+1}}\sin(2^n\pi x)\,dx\right]=0$$

$$\int_0^1 f(x,0)\,dx=0$$

$$\therefore \int_0^1\int_0^1 f(x,y)\,dx\,dy=\int_0^1 0\,dy=0$$

For each $0<x<\frac{1}{2}$, $\exists\, n\in\mathbb{N}$ s.t. $x\in[2^{-n-1},2^{-n}]$,

$$\int_0^1 f(x,y)\,dy$$
$$=\sin(2^{n+1}\pi x)\left[2^{2n+2}\int_{2^{-n-1}}^{2^{-n}}\sin(2^{n+1}\pi y)\,dy-2^{2n+1}\int_{2^{-n}}^{2^{-n+1}}\sin(2^n\pi y)\,dy\right]=0.$$

$$\int_0^1 f(0,y)\,dy=0$$

For each $\frac{1}{2}\leq x\leq 1$,

$$\int_0^1 f(x,y)\,dy=\int_{\frac{1}{2}}^1 2^2\sin(2\pi x)\sin(2\pi y)\,dy=\frac{-4\sin(2\pi x)}{\pi}$$

$$\therefore \int_0^1\int_0^1 f(x,y)\,dy\,dx=\int_{\frac{1}{2}}^1\frac{-4\sin(2\pi x)}{\pi}\,dx=\frac{4}{\pi^2}\neq 0=\int_0^1\int_0^1 f(x,y)\,dx\,dy$$

i.e. The result of Fubini's Thm doesn't hold.

3.12.5

#6 (a). Define $f: R \to \mathbb{R}$ by $f(x) = f_1(x_1) f_2(x_2) \cdots f_n(x_n)$ for $x = (x_1, \cdots, x_n) \in R$.

f_k is integrable on $[a_k, b_k]$ for $k = 1, \cdots, n$

$\Rightarrow f_k$ is bounded on $[a_k, b_k]$ for $k = 1, \cdots, n$.

$\Rightarrow \exists M > 1$ s.t. $|f_k(x_k)| \le M \quad \forall x_k \in [a_k, b_k]$ for $k = 1, \cdots, n$.

For any rectangle $I = [c_1, d_1] \times [c_2, d_2] \times \cdots \times [c_n, d_n] \subset R$

We can show that

$$\sup_{x \in I} f(x) - \inf_{x \in I} f(x) \le 4^n M^n \sum_{k=1}^{n} \left(\sup_{x_k \in [c_k, d_k]} f_k(x_k) - \inf_{x_k \in [c_k, d_k]} f_k(x_k) \right)$$

by induction. (Show it!)

Hence for each grid $\mathcal{G}$ on R,

$$U(f, \mathcal{G}) - L(f, \mathcal{G}) \le 4^n M^n \sum_{k=1}^{n} \left[U(f_k, P_k(\mathcal{G})) - L(f_k, P_k(\mathcal{G})) \right] \prod_{\substack{j=1 \\ j \ne k}}^{n} (b_j - a_j)$$

where $P_k(\mathcal{G})$ is defined as in P. 381.

For each $\varepsilon > 0$, choose P_k the partion of $[a_k, b_k]$ s.t.

$$U(f_k, P_k) - L(f_k, P_k) < \left(4^n M^n n \prod_{\substack{j=1 \\ j \ne k}}^{n} (b_j - a_j) \right)^{-1} \varepsilon .$$

Then $U(f, \mathcal{G}) - L(f, \mathcal{G}) < \varepsilon$ where $\mathcal{G}$ is the grid associated to $P_1, P_2, \cdots, P_n$. i.e. f is integrable on R.

According to Lemma 12.36 and the previous result, f is integrable on R, and for each $(x_1, x_2, \cdots, x_{n-1}) \in [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}]$, $f_1(x_1) f_2(x_2) \cdots f_{n-1}(x_{n-1}) f_n(\cdot)$ is integrable on $[a_n, b_n]$ $\Rightarrow$

$$\int_{a_n}^{b_n} f_1(x_1) f_2(x_2) \cdots f_{n-1}(x_{n-1}) f_n(x_n) \, dx_n = f_1(x_1) f_2(x_2) \cdots f_{n-1}(x_{n-1}) \int_{a_n}^{b_n} f_n(x_n) \, dx_n$$

is integrable on $[a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}]$, and

$$\int_R f(x) \, dx = \int_{[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}]} f_1(x_1) f_2(x_2) \cdots f_{n-1}(x_{n-1}) \int_{a_n}^{b_n} f_n(x_n) \, dx_n \, d(x_1, \cdots, x_{n-1})$$

$$= \left(\int_{[a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}]} f_1(x_1) f_2(x_2) \cdots f_{n-1}(x_{n-1}) \, d(x_1, \cdots, x_{n-1}) \right) \left(\int_{a_n}^{b_n} f_n(x_n) \, dx_n \right)$$

Apply the previous result to $n-1, n-2, \cdots, 1$, we get

$$\int_R f(x) \, dx = \left(\int_{[a_1, b_1] \times \cdots \times [a_{n-2}, b_{n-2}]} f_1(x_1) f_2(x_2) \cdots f_{n-2}(x_{n-2}) \, d(x_1, \cdots, x_{n-2}) \right) \cdot \left(\int_{a_{n-1}}^{b_{n-1}} f_{n-1}(x_{n-1}) \, dx_{n-1} \right) \left(\int_{a_n}^{b_n} f_n(x_n) \, dx_n \right)$$

$$= \cdots\cdots$$

$$= \left(\int_{a_1}^{b_1} f_1(x_1)\,dx_1\right)\left(\int_{a_2}^{b_2} f_2(x_2)\,dx_2\right)\cdots\left(\int_{a_n}^{b_n} f_n(x_n)\,dx_n\right)$$

#6(b) Define $f_k : [0,1] \longrightarrow \mathbb{R}$ by $f_k(x_k) = e^{-x_k}$ for $x_k \in [0,1]$ for $k=1,\dots,n$.

Then f_k is integrable on $[0,1]$.

By (a),

$$\int_Q e^{-\mathbf{x}\cdot\mathbf{y}}\,d\mathbf{x} = \int_Q e^{-x_1} e^{-x_2} \cdots e^{-x_n}\,d\mathbf{x}$$

$$= \left(\int_0^1 e^{-x_1}\,dx_1\right)\left(\int_0^1 e^{-x_2}\,dx_2\right)\cdots\cdots\left(\int_0^1 e^{-x_n}\,dx_n\right)$$

$$= (1-e^{-1})^n = \left(\frac{e-1}{e}\right)^n.$$

P.S. In (a), in fact,

$$\sup_{\mathbf{x}\in I} f(\mathbf{x}) - \inf_{\mathbf{x}\in I} f(\mathbf{x})$$

$$\leq \prod_{k=1}^{n}(M_k - m_k) + M\sum_{\lambda_1=1}^{n}\prod_{\substack{k=1\\k\neq\lambda_1}}^{n}(M_k-m_k) + M^2\sum_{1\leq\lambda_1<\lambda_2\leq n}\prod_{\substack{k=1\\k\neq\lambda_1,\lambda_2}}^{n}(M_k-m_k)$$

$$+ M^3\sum_{1\leq\lambda_1<\lambda_2<\lambda_3\leq n}\prod_{\substack{k=1\\k\neq\lambda_1,\lambda_2,\lambda_3}}^{n}(M_k-m_k) + \cdots\cdots + M^{n-1}\sum_{k=1}^{n}(M_k-m_k)$$

where $M_k = \sup_{x_k\in[c_k,d_k]} f_k(x_k)$, $m_k = \inf_{x_k\in[c_k,d_k]} f_k(x_k)$.