§11.5

#2 $f \in C'(B_r(0)) \Rightarrow f$ is conti. on $B_r(0)$

In particular, f is conti. at 0.

$\therefore |f(0)| = |\lim_{x\to 0} f(x)| = |\lim_{k\to\infty} f(x_k)| \le \lim_{k\to\infty} \|x_k\|^\alpha = 0$

$\therefore f(0) = 0$

$\bar{E} \subset B_r(0) \Rightarrow \bar{E}$ is compact $\Rightarrow \|x\| \le \|x_0\| \equiv r_0 < r \quad \forall x \in E$ for some $x_0 \in E$.

$\therefore \bar{E} \subset \overline{B_{r_0}(0)}$.

$f \in C'(\overline{B_{r_0}(0)})$ & $\overline{B_{r_0}(0)}$ is compact $\Rightarrow \|Df(x)\| \le M \quad \forall x \in \overline{B_{r_0}(0)}$ for some $M > 0$.

By Taylor's formula on $\mathbb{R}^n$, for each $x \in E$, $\exists\, c = c(x) \in L(x;0) \subseteq \overline{B_{r_0}(0)}$

s.t.

$$\underbrace{f(x) - f(0)}_{= f(x)} = D^{(1)} f(c; x-0) = Df(c) \cdot x$$

$\therefore \quad |f(x)| \le \|Df(c)\| \|x\| \le M \|x\|$.

#4. It's easy to show that

$$D^{(k)} f(a; h) = \sum_{j=0}^{k} \binom{k}{j} \frac{\partial^k f}{\partial x^j \partial y^{k-j}}(a)\, h_1^j h_2^{k-j}$$

by induction on k.

Then, just apply Taylor's formula on $\mathbb{R}^n$ with $V = B_r(x_0, y_0)$, $a = (x_0, y_0)$, $x = (x, y)$, $c = (c, d)$.

§11.5

#5(a). Note that

$$[(tx+(1-t)a)-a]^2+(y-b)^2$$
$$= t^2(x-a)^2+(y-b)^2 \le (x-a)^2+(y-b)^2 < r \quad \text{for } t\in[0,1].$$

$\Rightarrow (tx+(1-t)a,\ y)\in B_r(a,b)$ for $t\in[0,1]$.

Similarly, $(a,\ ty+(1-t)b)\in B_r(a,b)$ for $t\in[0,1]$.

By Chain rule, g is differentiable for $0<t<1$, and

$$g'(x) = f_x(tx+(1-t)a,\ y)(x-a)+f_y(a,\ ty+(1-t)b)(y-b).$$

(b). f is differentiable on $B_r(a,b) \Rightarrow f$ is conti. on $B_r(a,b)$

$\Rightarrow g$ is conti. on $[0,1]$.

By (a), g is differentiable on $(0,1)$

Then by mean value thm, $\exists\ \theta\in(0,1)$ s.t.

$$g(1)-g(0) = g'(\theta)$$
$$= f_x(\underbrace{\theta x+(1-\theta)a}_{\substack{\parallel \\ c \text{ between } a\ \&\ x}},\ y)(x-a)+f_y(a,\underbrace{\theta y+(1-\theta)b}_{\substack{\parallel \\ d \text{ between } b\ \&\ y}})(y-b)$$

$$g(1)-g(0) = f(x,y)+f(a,y)-f(a,y)-f(a,b) = f(x,y)-f(a,b).$$

#6. As in the proof of Thm 11.37, let $F(t)=f(a+th)$, then

$$F^{(k)}(0) = D^{(k)}f(a;h)$$

and

$$F^{(p)}(t) = D^{(p)}f(a+th;h)$$

for $k=1,2,\cdots,p-1$, $t\in I_\delta$.

Then by Thm 7.52,

$$F(1)-F(0) = \sum_{k=1}^{p-1}\frac{F^{(k)}(0)}{k!}+\frac{1}{(p-1)!}\int_0^1(1-t)^{p-1}F^{(p)}(t)\,dt$$

$$f(x)-f(a) = F(1)-F(0) = \sum_{k=1}^{p-1}\frac{1}{k!}D^{(k)}f(a;h)+\frac{1}{(p-1)!}\int_0^1(1-t)^{p-1}D^{(p)}f(a+th;h)\,dt$$

311.5

#8. V is open, $(a,b)\in V \Rightarrow \exists R>0$ s.t. $\overline{B_R(a,b)} \subset V$.

$\Rightarrow (a+r\cos\theta,\ b+r\sin\theta) \in \overline{B_R(a,b)} \subset V \quad \forall r\in[0,R],\ \theta\in[0,2\pi]$.

Since $f\in C^3(V)$ and $\overline{B_R(a,b)}$ is compact, there exists $M>0$ s.t. all third order partial derivatives of f are bounded by M on $\overline{B_R(a,b)}$.

$f\in C^3(V)$. Apply Taylor's formula on $\mathbb{R}^n$, for each $r\in[0,R]$, $\theta\in[0,2\pi]$

$\exists\ (c,d) \in L\big((a+r\cos\theta,\ b+r\sin\theta);\ (a,b)\big)$ s.t.

$$\begin{aligned} &f(a+r\cos\theta,\ b+r\sin\theta) \\ &= f(a,b) + f_x(a,b)r\cos\theta + f_y(a,b)r\sin\theta + \tfrac{1}{2}f_{xx}(a,b)r^2\cos^2\theta + f_{xy}(a,b)r^2\cos\theta\sin\theta \\ &\quad + \tfrac{1}{2}f_{yy}(a,b)r^2\sin^2\theta + \tfrac{1}{6}f_{xxx}(c,d)r^3\cos^3\theta + \tfrac{1}{2}f_{xxy}(c,d)r^3\cos^2\theta\sin\theta \\ &\quad + \tfrac{1}{2}f_{xyy}(c,d)r^3\cos\theta\sin^2\theta + \tfrac{1}{6}f_{yyy}(c,d)r^3\sin^3\theta \end{aligned}$$

Note that

$$\int_0^{2\pi}\cos 2\theta\, d\theta = \int_0^{2\pi}\cos\theta\cos 2\theta\, d\theta = \int_0^{2\pi}\sin\theta\cos 2\theta\, d\theta = \int_0^{2\pi}\cos\theta\sin\theta\cos 2\theta\, d\theta = 0$$

and $\displaystyle\int_0^{2\pi}\cos^2\theta\cos 2\theta\, d\theta = \frac{\pi}{2}$, $\displaystyle\int_0^{2\pi}\sin^2\theta\cos 2\theta\, d\theta = -\frac{\pi}{2}$

$$\begin{aligned} \therefore\ &\frac{4}{\pi r^2}\int_0^{2\pi} f(a+r\cos\theta,\ b+r\sin\theta)\cos 2\theta\, d\theta \\ &= f_{xx}(a,b) - f_{yy}(a,b) + \frac{4r}{\pi}\Big\{\int_0^{2\pi}\Big[\tfrac{1}{6}f_{xxx}(c,d)\cos^3\theta + \tfrac{1}{2}f_{xxy}(c,d)\cos^2\theta\sin\theta \\ &\qquad + \tfrac{1}{2}f_{xyy}(c,d)\cos\theta\sin^2\theta + \tfrac{1}{6}f_{yyy}(c,d)\sin^3\theta\Big]\cos 2\theta\, d\theta \end{aligned}$$

But $\displaystyle\left|\frac{4r}{\pi}\Big\{\int_0^{2\pi}\Big[\tfrac{1}{6}f_{xxx}(c,d)\cos^3\theta + \tfrac{1}{2}f_{xxy}(c,d)\cos^2\theta\sin\theta + \tfrac{1}{2}f_{xxy}(c,d)\cos\theta\sin^2\theta + \tfrac{1}{6}f_{yyy}(c,d)\sin^3\theta\Big]\cos 2\theta\, d\theta\right.$

$$\le \frac{4r}{\pi}\int_0^{2\pi}\Big(\frac{M}{6} + \frac{M}{2} + \frac{M}{2} + \frac{M}{6}\Big)d\theta = \frac{16M}{3\pi}r \longrightarrow 0 \quad \text{as} \quad r\to 0$$

$$\therefore \lim_{r\to 0}\frac{4}{\pi r^2}\int_0^{2\pi} f(a+r\cos\theta,\ b+r\sin\theta)\cos 2\theta\, d\theta = f_{xx}(a,b) - f_{yy}(a,b).$$

§11.6

#4. Let $F_1(u,v,x,y) = xu^2 + yv^2 + xy - 9$

$F_2(u,v,x,y) = xv^2 + yu^2 - xy - 7$

$$\frac{\partial(F_1,F_2)}{\partial(u,v)} = \det\begin{bmatrix} 2xu & 2yv \\ 2yu & 2xv \end{bmatrix} = 4uv(x^2-y^2)$$

At (x_0, y_0, u_0, v_0) s.t. $4u_0v_0(x_0^2-y_0^2) \neq 0$ and $(F_1,F_2)(x_0,y_0,u_0,v_0) = (0,0)$, The Implicit Function Thm can be applied. i.e. u, v can be represented as C^1 functions of x, y near (x_0, y_0) with $(F_1,F_2)(x,y,u(x,y),v(x,y)) = 0$

i.e.

$$\begin{aligned} x\,u(x,y)^2 + y\,v(x,y)^2 + xy &= 9 \\ x\,v(x,y)^2 + y\,u(x,y)^2 - xy &= 7 \end{aligned} \qquad \text{near } (x_0, y_0)$$

Add up above two equations, we get

$$(x+y)\left[u(x,y)^2 + v(x,y)^2\right] = 16 \qquad \text{near } (x_0,y_0)$$

$$\Rightarrow \quad u(x,y)^2 + v(x,y)^2 = \frac{16}{x+y}$$

#5. Define $F = (F_1,F_2,F_3,F_4): \mathbb{R}^6 \to \mathbb{R}^4$ by

$F_1(x,y,u,v,s,t) = u^2 + sx + ty$, $F_2(x,y,u,v,s,t) = v^2 + tx + sy$

$F_3(x,y,u,v,s,t) = 2s^2x + 2t^2y - 1$, $F_4(x,y,u,v,s,t) = s^2x - t^2y$.

$$\frac{\partial(F_1,F_2,F_3,F_4)}{\partial(u,v,s,t)}(x_0,y_0,u_0,v_0,s_0,t_0) = \det\begin{bmatrix} 2u_0 & 0 & x_0 & y_0 \\ 0 & 2v_0 & y_0 & x_0 \\ 0 & 0 & 4s_0x_0 & 4t_0y_0 \\ 0 & 0 & 2s_0x_0 & -2t_0y_0 \end{bmatrix} = -32u_0v_0x_0y_0s_0t_0 \neq 0$$

$\therefore$ By Implicit Function Thm, $\exists$ an open ball B containing (x_0,y_0) s.t. u, v, s, t are C^1 and satisfy $(*)$ on B, and $u(x_0,y_0) = u_0$, $v(x_0,y_0) = v_0$, $s(x_0,y_0) = s_0$, $t(x_0,y_0) = t_0$.

§11.6

#6. (a). 1-1: Suppose $(x_1, y_1), (x_2, y_2) \in E$ s.t. $f(x_1, y_1) = f(x_2, y_2)$

i.e. $\begin{cases} x_1 + y_1 = x_2 + y_2 \\ x_1 y_1 = x_2 y_2 = x_2(x_1 + y_1 - x_2) \end{cases}$

$\Rightarrow (x_1 - x_2)(y_1 - x_2) = 0$

$\Rightarrow x_1 = x_2 \vee y_1 = x_2$

$\Rightarrow \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases} \vee \begin{cases} y_1 = x_2 > y_2 = x_1 > y_1 \quad (不合) \\ y_2 = x_1 \end{cases}$

$\Rightarrow (x_1, y_1) = (x_2, y_2)$ i.e. f is 1-1.

Onto: For each $(x, y) \in E$, $f(x, y) = (x+y, xy)$

$\begin{cases} x + y > 2\sqrt{xy} \\ xy > 0 \end{cases}$

$\therefore f(x, y) \in \{(s,t) : s > 2\sqrt{t}, t > 0\} := R.$

For each $(s,t) \in R$, let $x = \dfrac{s + \sqrt{s^2 - 4t}}{2}$, $y = \dfrac{s - \sqrt{s^2-4t}}{2}$.

Note that $(s,t) \in R$ $\therefore s^2 > s^2 - 4t > 0$

$\therefore x > y > 0$ i.e. $(x, y) \in E$.

Moreover, by simple computation, $f(x, y) = (s, t)$.

$\therefore f$ is 1-1 from E onto R, and $f^{-1}(s,t) = \left(\dfrac{s+\sqrt{s^2-4t}}{2}, \dfrac{s - \sqrt{s^2-4t}}{2}\right)$.

(b). $Df(x,y) = \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix}$. For $(x,y) \in E$, $\det(Df(x,y)) = x - y > 0$

$\therefore Df(x,y)$ is invertible on E and $[Df(x,y)]^{-1} = \dfrac{1}{x-y}\begin{pmatrix} x & -1 \\ -y & 1 \end{pmatrix}$

$I = D(f^{-1})(f(x,y))\, Df(x,y)$. $\therefore D(f^{-1})(f(x,y)) = [Df(x,y)]^{-1} = \dfrac{1}{x-y}\begin{pmatrix} x & -1 \\ -y & 1 \end{pmatrix}$

(c). $$D(f^{-1})(s,t) = \begin{bmatrix} \frac{\partial}{\partial s}\left(\frac{s+\sqrt{s^2-4t}}{2}\right) & \frac{\partial}{\partial t}\left(\frac{s+\sqrt{s^2-4t}}{2}\right) \\ \frac{\partial}{\partial s}\left(\frac{s-\sqrt{s^2-4t}}{2}\right) & \frac{\partial}{\partial t}\left(\frac{s-\sqrt{s^2-4t}}{2}\right) \end{bmatrix} = \begin{bmatrix} \frac{s+\sqrt{s^2-4t}}{2\sqrt{s^2-4t}} & \frac{-1}{\sqrt{s^2-4t}} \\ \frac{-s+\sqrt{s^2-4t}}{2\sqrt{s^2-4t}} & \frac{1}{\sqrt{s^2-4t}} \end{bmatrix}$$

By (b), $$D(f^{-1})(s,t) = \frac{1}{\frac{s+\sqrt{s^2-4t}}{2} - \frac{s-\sqrt{s^2-4t}}{2}} \begin{bmatrix} \frac{s+\sqrt{s^2-4t}}{2} & -1 \\ \frac{s-\sqrt{s^2-4t}}{2} & 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{s^2-4t}} \begin{bmatrix} \frac{s+\sqrt{s^2-4t}}{2\sqrt{s^2-4t}} & -1 \\ \frac{-s+\sqrt{s^2-4t}}{2\sqrt{s^2-4t}} & 1 \end{bmatrix}$$

§11.6

#9. (a). $\nabla f(t_0) = \begin{pmatrix} u'(t_0) \\ v'(t_0) \end{pmatrix} \neq 0 \Longleftrightarrow u'(t_0) \neq 0 \vee v'(t_0) \neq 0.$

(b). By (a), $u'(t_0) \neq 0$ or $v'(t_0) \neq 0$.

Case $u'(t_0) \neq 0$:

Define $F: \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $F(t, x) = u(t) - x$.

Then $\frac{\partial F}{\partial t}(t_0, x_0) = u'(t_0) \neq 0$, $F(t_0, x_0) = u(t_0) - x_0 = 0$

Hence by Implicit Function Thm, $\exists$ an open set $W \ni x_0$,

a unique C^1 function $g: W \longrightarrow \mathbb{R}$ s.t. $g(x_0) = t_0$

and $F(g(x), x) = u(g(x)) - x = 0 \quad \forall x \in W$. (i.e. $u(g(x)) = x$).

Case $v'(t_0) \neq 0$: Similar to the previous case.

§11.7

#5. By considering $-f$ if necessary, we may assume that $f_{xy}(a,b) \equiv M > 0$.

f_{xx}, f_{xy}, f_{yy} are conti. at (a,b)

$\Rightarrow \exists\ r>0$ s.t. $B_r(a,b) \subset V$ and

$$|f_{xx}| < \frac{M}{4}, \quad |f_{yy}| < \frac{M}{4}, \quad f_{xy} > \frac{M}{2} \quad \text{on } B_r(a,b) \equiv B$$

Note that f has all 2^{nd}-order partial derivative $\Rightarrow$ $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$ are conti. and differentiable w.r.t. x for each y fixed, w.r.t. y for each x fixed on V.

$\therefore$ By MVT, for each $x<a$, $(x,b)\in B$,

$$f_x(a,b) - f_x(x,b) = f_{xx}(c_1(x), b)(a-x) \quad \text{for some } c_1(x) \in (x,a)$$
$$/\!/ \qquad < \frac{M}{4}(a-x) \quad (\because (c_1(x), b) \in B)$$
$$-f_x(x,b)$$

$\therefore$ $f_x(x,b) > -\frac{M}{4}(a-x)$ for each $x<a$, $(x,b)\in B$. —— ①

Again by MVT, for each $x<a$, $y>b$, $(x,y)\in B$,

$$f_x(x,y) - f_x(x,b) = f_{xy}(x, c_2(y))(y-b) \quad \text{for some } c_2(y) \in (b,y)$$
$$> \frac{M}{2}(y-b) \quad (\because (x, c_2(y)) \in B).$$

$\Rightarrow$ By ①, $f_x(x,y) > f_x(x,b) + \frac{M}{2}(y-b) > -\frac{M}{4}(a-x) + \frac{M}{2}(y-b)$

for each $x<a$, $y>b$, $(x,y)\in B$ —— ②

MVT $\Rightarrow$ for each $y>b$, $(a,y)\in B$,

$$-\frac{M}{4}(y-b) < f_y(a,y) - f_y(a,b) = f_{yy}(a, c_3(y))(y-b) \quad \text{for some } c_3(y) \in (b,y)$$
$$f_y(a,y) \;/\!/ \qquad < \frac{M}{4}(y-b) \quad ((a, c_3(y)) \in B)$$

$\Rightarrow -\frac{M}{4}(y-b) < f_y(a,y) < \frac{M}{4}(y-b)$ $\forall\, y>b$, $(a,y)\in B$. —— ③

MVT $\Rightarrow$ for each $y>b$, $(a,y)\in B$,

$$f(a,y) - f(a,b) = f_y(a, c_4(y))(y-b) \quad \text{for some } c_4(y) \in (b,y)$$

$\Rightarrow$ By ③ $-\frac{M}{4}(y-b)^2 + f(a,b) < f(a,y) < f(a,b) + \frac{M}{4}(y-b)^2$ for each $y>b$, $(a,y)\in B$ —— ④

For $(x,y)\in B$ with $x<a$, $y>b$, MVT $\Rightarrow$

$$f(a,y)-f(x,y)=f_x(c_5(x),y)(a-x) \quad \text{for some } c_5(x)\in(x,a)$$

$$> -\frac{M}{4}(a-x)^2+\frac{M}{2}(a-x)(y-b) \quad \text{by ②}$$

(by ④) $\wedge$

$$f(a,b)+\frac{M}{4}(y-b)^2-f(x,y)$$

$$\Rightarrow f(x,y)<f(a,b)+\frac{M}{4}\left[(y-b)-(a-x)\right]^2 \quad \text{for each } (x,y)\in B,\ x<a,\ y>b$$

Let $(x_k, y_k)=\left(a-\frac{1}{k}, b+\frac{1}{k}\right)$ for each $k\in\mathbb{N}$ and $\frac{\sqrt{2}}{k}<r$,

then $(x_k,y_k)\in B$, $x_k<a$, $y_k>b$, hence

$$f(x_k,y_k)<f(a,b)+\frac{M}{4}\left[\frac{1}{k}-\frac{1}{k}\right]^2=f(a,b)$$

and $(x_k,y_k)\longrightarrow(a,b)$ as $k\longrightarrow\infty$.

$\therefore$ (a,b) is not a local minimum of f.

Similarly, by MVT, we can show that

$$f_x(x,b)>f_x(a,b)-\frac{M}{4}(x-a)=-\frac{M}{4}(x-a) \quad \text{for } x>a,\ (x,b)\in B.$$

$$\Rightarrow f_x(x,y)>f_x(x,b)+\frac{M}{2}(y-b)>-\frac{M}{4}(x-a)+\frac{M}{2}(y-b) \quad \text{for } x>a, y>b, (x,y)\in B$$

$$\Rightarrow f(x,y)>f(a,y)-\frac{M}{4}(x-a)^2+\frac{M}{2}(x-a)(y-b)$$

$$>f(a,b)-\frac{M}{4}(y-b)^2-\frac{M}{4}(x-a)^2+\frac{M}{2}(x-a)(y-b)$$

$$=f(a,b)-\frac{M}{4}\left[(y-b)-(x-a)\right]^2 \quad \text{for } x>a, y>b, (x,y)\in B.$$

Choose $(\tilde{x}_k,\tilde{y}_k)=\left(a+\frac{1}{k}, b+\frac{1}{k}\right)$ for each $k\in\mathbb{N}$ and $\frac{\sqrt{2}}{k}<r$,

then $(\tilde{x}_k,\tilde{y}_k)\in B$, $\tilde{x}_k>a$, $\tilde{y}_k>b$, $(\tilde{x}_k,\tilde{y}_k)\longrightarrow(a,b)$ as $k\longrightarrow\infty$,

and $f(\tilde{x}_k,\tilde{y}_k)>f(a,b)-\frac{M}{4}\left[\frac{1}{k}-\frac{1}{k}\right]^2=f(a,b)$

$\Rightarrow$ (a,b) is not a local maximum of f.

$\therefore$ (a,b) is a saddle point of f.

§11.7

#7 (a) Case $cD>0$: $DE>0 \Rightarrow cE>0$

$DE>0 \Rightarrow cE>0$

Subject the constraint $z = Dx^2+Ey^2$,

$$ax+by+cz = ax+by+cDx^2+cEy^2$$

$$\longrightarrow \infty \quad \text{as} \quad (x,y)\longrightarrow \infty.$$

Hence there are no maxima.

$$\begin{aligned} ax+by+cz &= ax+by+cDx^2+cEy^2 \\ &= cD\left(x^2-\frac{a}{cD}x+\frac{a^2}{4c^2D^2}\right)+cE\left(y^2-\frac{b}{cE}y+\frac{b^2}{4c^2E^2}\right) \\ &\quad -\frac{a^2}{4cD}-\frac{b^2}{4cE} \\ &= cD\left(x-\frac{a}{2cD}\right)^2+cE\left(y-\frac{b}{2cE}\right)^2-\frac{a^2}{4cD}-\frac{b^2}{4cE} \\ &\geq -\frac{a^2}{4cD}-\frac{b^2}{4cE} \end{aligned}$$

Therefore, there exists just one minimum at $\left(\frac{a}{2cD}, \frac{b}{2cE}\right)$.

Case $cD<0$:

Just consider the problem $-ax-by-cz$ subject to the constraint $z=Dx^2+Ey^2$ to get a minimum, which implies $ax+by+cz$ has a maximum at $\left(\frac{a}{2cD}, \frac{b}{2cE}\right)$.

(b). When $DE<0$, subject to the constraint $z=Dx^2+Ey^2$,

$$ax+by+cz = ax+by+cDx^2+cEy^2$$

$$\begin{cases} \longrightarrow (\operatorname{sgn} cD)\,\infty & \text{as } x\to\infty,\ y=0 \\ \longrightarrow (\operatorname{sgn} cE)\,\infty & \text{as } y\to\infty,\ x=0. \end{cases}$$

But $(\operatorname{sgn} cD)(\operatorname{sgn} cE)<0$ $\quad (\because DE<0)$

$\therefore$ There are no extrema.

§11.7

#8. (a). The Statement is false.

For example, define $g: \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$g(x,y,z) = \begin{cases} x^2+y^2+z^2+x & \text{if } x^2+y^2+z^2+x \neq 0 \\ & \text{or } x=y=z=0 \\ -x^2-y^2-z^2+x & \text{otherwise.} \end{cases}$$

Then g is differentiable at $(0,0,0)$, $Dg(0,0,0)=(1,0,0)$ & $g(x,y,z)=0 \Leftrightarrow (x,y,z)=(0,0,0)$. (Check it!)

Hence, there is only one point $(0,0,0)$ s.t. $g(x,y,z)=0$.

$\therefore$ Subject to the constraint $g(x,y,z)=0$, f has an extremum at $(0,0,0)$ for any f. Just choose f s.t. $Df(0,0,0) \nparallel (1,0,0)$, then it's a counter example.

Choose $f(x,y,z)=z$, f is diff on $\mathbb{R}^3$ and $Df(0,0,0) = (0,0,1)$.

$$\frac{\partial f}{\partial x}(0,0,0)\,\frac{\partial g}{\partial z}(0,0,0) - \frac{\partial f}{\partial z}(0,0,0)\,\frac{\partial g}{\partial x}(0,0,0) = -1 \neq 0.$$

(b). For each $n \in \mathbb{N}$, let $\mathcal{X}_n = (n, n, \frac{16}{n^2})$, $\mathcal{Y}_n = (n, -n, -\frac{16}{n^2})$,

then $\mathcal{X}_n$, $\mathcal{Y}_n$ are on $xyz=16$ but

$$f(\mathcal{X}_n) = 4n^2 + \frac{64}{n} \geq 4n^2 \longrightarrow \infty \quad \text{as } n \longrightarrow \infty$$

$$f(\mathcal{Y}_n) = -4n^2 \longrightarrow -\infty \quad \text{as } n \longrightarrow \infty$$

Hence, there are no absolute extrema of f subject to the constraint $xyz=16$.

But by Langrange multiplier, it's easy to find out that there is a local minimum of f subject to $xyz=16$ at $(2,2,4)$.