§11.2

#3. $$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{h^2}{\sin h}}{h}=\lim_{h\to 0}\frac{h}{\sin h}=1.$$

$$f_y(0,0)=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to 0}\frac{\frac{k^2}{\sin k}}{k}=\lim_{k\to 0}\frac{k}{\sin k}=1.$$

$$\frac{f(h,k)-f(0,0)-f_x(0,0)h-f_y(0,0)k}{\sqrt{h^2+k^2}}=\frac{\sqrt{h^2+k^2}}{\sin\sqrt{h^2+k^2}}-\frac{h+k}{\sqrt{h^2+k^2}}.$$

Note that $\displaystyle\lim_{(h,k)\to(0,0)}\frac{\sqrt{h^2+k^2}}{\sin\sqrt{h^2+k^2}}=1.$

But $\displaystyle\lim_{(h,k)\to(0,0)}\frac{h+k}{\sqrt{h^2+k^2}}$ doesn't exist. $\left(\begin{array}{l}\text{Both } (\frac{1}{n},0),\ (\frac{1}{n},\frac{1}{n})\to(0,0)\\ \text{But } \dfrac{\frac{1}{n}+0}{\sqrt{\frac{1}{n^2}+0}}\to 1 \quad \text{as } n\to\infty\\ \quad\ \dfrac{\frac{1}{n}+\frac{1}{n}}{\sqrt{\frac{1}{n^2}+\frac{1}{n^2}}}\to\sqrt{2}\end{array}\right)$

$\therefore$ The limit of $\displaystyle\frac{f(h,k)-f(0,0)-\nabla f(0,0)\cdot(h,k)}{\sqrt{h^2+k^2}}$ doesn't exist

Hence f is not differentiable at $(0,0)$.

#4. $r>0$, $f: B_r(0) \longrightarrow \mathbb{R}$. $|f(x)| \le \|x\|^\alpha$ for some $\alpha>1$ $\forall x \in B_r(0)$

(a). $0 \le |f(0)| \le \|0\|^\alpha = 0 \Rightarrow f(0)=0$.

$$0 \le \frac{|f(h)-f(0)|}{\|h\|} = \frac{|f(h)|}{\|h\|} \le \frac{\|h\|^\alpha}{\|h\|} = \|h\|^{\alpha-1} \quad \forall h \in B_r(0)$$

$$\|h\|^{\alpha-1} \to 0 \text{ as } h \to 0$$

$$\Rightarrow \lim_{h\to 0} \frac{|f(h)-f(0)|}{\|h\|} = 0$$

$\therefore$ f is differentiable at 0 with $Df(0)=0$.

(b). The differentiability of f at $(0,0)$ may be true or not.

For example,

① $f(x) = a\cdot x$ for some $a \in \mathbb{R}^n$ with $\|a\|=1$.

Then $|f(x)| \le \|a\|\|x\| = \|x\|$.

And $$\frac{|f(h)-f(0)-a\cdot h|}{\|h\|} = 0 \longrightarrow 0 \text{ as } h \to 0$$

$\therefore$ f is differentiable at $(0,0)$ with $Df(0)=a$.

② $f(x) = \|x\|$

$$|f(x)| = \|x\|$$

But $$\frac{f(he_1)-f(0)}{h} = \frac{\|he_1\|}{h} = \frac{|h|}{h} = \begin{cases} 1 & \text{if } h>0 \\ -1 & \text{if } h<0 \end{cases}$$

$\therefore$ $f_{x_1}(0)$ doesn't exists

$\Rightarrow$ f is not differentiable at $(0,0)$.

#5.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} 0 = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} 0 = 0.$$

$$0 \le \left| \frac{f(h,k) - f(0,0) - \nabla f(0,0)\cdot(h,k)}{\sqrt{h^2+k^2}} \right| = \left| \frac{|hk|^\alpha \log(h^2+k^2)}{\sqrt{h^2+k^2}} \right|$$

$$\le \frac{\left(\frac{h^2+k^2}{2}\right)^\alpha \log(h^2+k^2)}{\sqrt{h^2+k^2}} = 2^{-\alpha}(h^2+k^2)^{\alpha-\frac{1}{2}} \log(h^2+k^2)$$

$\longrightarrow 0$ as $(h,k) \longrightarrow (0,0)$

$$\left(\because \lim_{r\to 0} r^{\alpha-\frac{1}{2}} \log r = \lim_{r\to 0} \frac{\log r}{r^{\frac{1}{2}-\alpha}} = \lim_{r\to 0} \frac{\frac{1}{r}}{(\frac{1}{2}-\alpha) r^{-\frac{1}{2}-\alpha}} \right.$$

$$\left. = \lim_{r\to 0} \frac{1}{\frac{1}{2}-\alpha} r^{\alpha-\frac{1}{2}} = 0 \quad \text{for } \alpha > \frac{1}{2} \right)$$

$$\therefore \lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-\nabla f(0,0)\cdot(h,k)}{\sqrt{h^2+k^2}} = 0$$

i.e. f is differentiable at $(0,0)$.

$\alpha < \frac{3}{2} \Leftrightarrow 3-2\alpha > 0$

#6. $f_x(x,y) = \frac{4x^3}{(x^2+y^2)^\alpha} - \frac{2\alpha x(x^4+y^4)}{(x^2+y^2)^{\alpha+1}} = \frac{(4-2\alpha)x^5+4x^3y^2-2\alpha xy^4}{(x^2+y^2)^{\alpha+1}}$

for $(x,y)\neq(0,0)$

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{h^4}{h^{2\alpha}}}{h} = \lim_{h\to 0} h^{3-2\alpha} = 0$$

$$\therefore f_x(x,y) = \begin{cases} \frac{(4-2\alpha)x^5+4x^3y^2-2\alpha xy^4}{(x^2+y^2)^{\alpha+1}} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$$

$$0 \le |f_x(x,y)| \le \frac{|4-2\alpha|(x^2+y^2)^{\frac{5}{2}} + 4(x^2+y^2)^{\frac{5}{2}} + 2|\alpha|(x^2+y^2)^{\frac{5}{2}}}{(x^2+y^2)^{\alpha+1}}$$

$$= \left(|4-2\alpha|+4+2|\alpha|\right)(x^2+y^2)^{\frac{3-2\alpha}{2}}$$

$$\longrightarrow 0 \quad \text{as} \quad (x,y)\longrightarrow(0,0)$$

$$\therefore \lim_{(x,y)\to(0,0)} f_x(x,y) = 0 = f_x(0,0)$$

i.e. f_x is conti. at $(0,0)$

Obviously f_x is conti. on $\mathbb{R}^2\setminus\{(0,0)\}$

$\therefore$ f_x is conti. in $\mathbb{R}^2$.

Similarly, f_y is conti in $\mathbb{R}^2$.

$\Rightarrow f \in C^1(\mathbb{R}^2)$

$\Rightarrow f$ is diffentiable in $\mathbb{R}^2$.

§11.3

#4. (a) f is differentiable at $a \Rightarrow f$ is conti. at a

$\therefore$ For $\frac{|f(a)|}{2} > 0$, $\exists\, \delta > 0$ s.t.

$$|f(x) - f(a)| < \frac{|f(a)|}{2} \quad \text{if } \|x-a\| < \delta$$

$$\vee\!\!/$$

$$|f(a)| - |f(x)|$$

$$\Rightarrow 0 < \frac{|f(a)|}{2} \le |f(x)| \quad \text{if} \quad \|x-a\| < \delta.$$

$\therefore$ If $\|h\| < \delta$, then $\|(a+h)-a\| = \|h\| < \delta$

$\Rightarrow \quad |f(a+h)| \geq \frac{|f(a)|}{2} > 0$ i.e. $f(a+h) \neq 0$.

(b). $$\frac{\|Df(a)(h)\|}{\|h\|} \le \frac{\|Df(a)\|\,\|h\|}{\|h\|} = \|Df(a)\| \quad \forall\, h \in \mathbb{R}^n \setminus \{0\}.$$

i.e. $\frac{Df(a)(h)}{\|h\|}$ is bdd. $\forall\, h \in \mathbb{R}^n \setminus \{0\}$.

(c). $$\frac{1}{f(a+h)} - \frac{1}{f(a)} - T(h) = \frac{1}{f(a+h)} - \frac{1}{f(a)} + \frac{Df(a)(h)}{f^2(a)}$$

$$= \frac{f(a) - f(a+h) + Df(a)(h)}{f(a)f(a+h)} - \frac{Df(a)(h)f(a) - Df(a)(h)f(a+h)}{f^2(a)f(a+h)}$$

$$= \frac{f(a) - f(a+h) + Df(a)(h)}{f(a)f(a+h)} + \frac{(f(a+h) - f(a))Df(a)(h)}{f^2(a)f(a+h)}$$

(d). $$\left| \frac{\frac{1}{f(a+h)} - \frac{1}{f(a)} - T(h)}{\|h\|} \right| \le \left| \frac{f(a+h) - f(a) - Df(a)(h)}{\|h\|} \right| \frac{1}{|f(a)||f(a+h)|}$$

$$\le \underbrace{\frac{|f(a+h) - f(a) - Df(a)(h)|}{\|h\|}}_{\downarrow\ 0 \text{ as } h\to 0} \underbrace{\frac{1}{|f(a)||f(a+h)|}}_{\downarrow\ \frac{1}{|f^2(a)|} \text{ as } h\to 0} + \underbrace{\frac{|f(a+h) - f(a)|}{|f^2(a)||f(a+h)|}}_{\downarrow\ 0 \text{ as } h\to 0} \underbrace{\frac{|Df(a)(h)|}{\|h\|}}_{\leq\ \|Df(a)\|}$$

$\longrightarrow 0$ as $h \to 0$

$\therefore \frac{1}{f}$ is differentiable at a & $D\left(\frac{1}{f}\right)(a) = -\frac{Df(a)}{f^2(a)}$.

§11.3

#4(e) 更正: f is a real-value vector function. (say $f: \mathbb{R}^n \to \mathbb{R}^n$)
g is a real-value scalar function.

f, g are differentiable at a & $g(a) \neq 0$.

pf: Obviously, a vector function $F = (F_1, F_2, \cdots, F_m)$ is differentiable at a if and only if F_i is differentiable at a for each $1 \le i \le m$.

Hence, it's sufficient to show that f_i is diff. at a and

$$D\left(\frac{f_i}{g}\right)(a) = \frac{g(a)Df_i(a) - f_i(a)Dg(a)}{g^2(a)}.$$

But by (d), $1/g$ is diff. at a, and

$$D\left(\frac{1}{g}\right)(a) = -\frac{Dg(a)}{g^2(a)}.$$

Inner product (but in $\mathbb{R}$, it is just the product of f_i & $1/g$)

Then by Thm 11.20, $\frac{f_i}{g} = f_i \cdot \left(\frac{1}{g}\right)$ is diff. at a and

$$D\left(\frac{f_i}{g}\right)(a) = \frac{1}{g(a)} Df_i(a) + f_i(a) D\left(\frac{1}{g}\right)(a)$$

$$= \frac{g(a)Df_i(a) - f_i(a)Dg(a)}{g^2(a)}$$

#

#5. (a) $f(x,y) = x^3 \sin y$

$\nabla f(x,y) = (3x^2 \sin y,\ x^3 \cos y)$ is conti.

$\therefore f \in C' \Rightarrow f$ is differentiable

$f(0,0) = 0$

$\nabla f(0,0) = (0,0)$

Tangent plane : $z = 0$

(b) $f(x,y) = x^3 y - x y^3$

$\nabla f(x,y) = (3x^2 y - y^3,\ x^3 - 3xy^2) \in C$

$\Rightarrow f \in C' \Rightarrow f$ is differentiable.

$f(1,1) = 0$

$\nabla f(1,1) = (2, -2)$

Tangent plane : $z = 2(x-1) - 2(y-1) = 2x - 2y$

#9. $w \in C' \Rightarrow w$ is differentiable $\qquad Dw(1,2,1) = (4,1,1)$

$\Delta w = w(1.01, 1.98, 1.03) - w(1,2,1)$

$\fallingdotseq 4(1.01-1) + 1(1.98-2) + 1\cdot(1.03-1)$

$= 0.04 - 0.02 + 0.03$

$= 0.05$

$\Delta w = w(1.01, 1.98, 1.03) - w(1,2,1)$

$= 3.049798 - 3$

$= 0.049798$

§11.4

#5. $u(x,t) = \dfrac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$

(a) $u_x(x,t) = -\dfrac{x}{2t}\dfrac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$

$u_{xx}(x,t) = -\dfrac{1}{2t}\dfrac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} + \dfrac{x^2}{4t^2}\dfrac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$

$u_t(x,t) = -\dfrac{1}{2\sqrt{4\pi t^3}}e^{-\frac{x^2}{4t}} + \dfrac{x^2}{4t^2}\dfrac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$

$= u_{xx}(x,t)$.

(b). $0 < u(x,t) = \dfrac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} \le \dfrac{e^{-\frac{a^2}{4t}}}{\sqrt{4\pi t}}$ for $x \in [a,\infty)$

$\lim\limits_{t\to 0}\dfrac{e^{-\frac{a^2}{4t}}}{\sqrt{4\pi t}} = \lim\limits_{t\to 0}\dfrac{\frac{1}{\sqrt{4\pi t}}}{e^{\frac{a^2}{4t}}} = \lim\limits_{t\to 0}\dfrac{-\frac{1}{2\sqrt{4\pi t^3}}}{-\frac{a^2}{4t^2}e^{\frac{a^2}{4t}}}$

$= \lim\limits_{t\to 0}\dfrac{\sqrt{t}}{\sqrt{\pi}e^{\frac{a^2}{4t}}} = 0$

$\therefore u(x,t) \to 0$ as $t \to 0+$ uniformly for $x \in [a,\infty)$

#6. $f: I \to \mathbb{R}^m$ is differentiable $(f = (f_1, f_2, \cdots, f_m))$.

$\Rightarrow f_i$ is differentiable for each $1 \le i \le m$.

$f(t) \in \partial B_r(0) \quad \forall t \in I$.

i.e. $\|f(t)\|^2 = r^2$

$\|f(t)\|^2 = \sum_{i=1}^m f_i^2(t)$

Differentiate w.r.t. $t \Rightarrow 2\sum_{i=1}^m f_i(t) f_i'(t) = 0 \quad \forall t \in I$

$2 f(t)\cdot f'(t) = 2\sum_{i=1}^m f_i(t) f_i'(t)$

$\therefore f(t)\cdot f'(t) = 0 \quad \forall t \in I$

#7. (a). f is differentiable at a

$$\therefore \lim_{h \to 0} \frac{|f(a+h) - f(a) - Df(a)\cdot h|}{\|h\|} = 0.$$

$$\therefore \lim_{t \to 0} \frac{|f(a+tu) - f(a) - Df(a)\cdot(tu)|}{\|tu\|} = 0$$

$$\| \quad \lim_{t \to 0} \left| \frac{f(a+tu) - f(a) - t\,Df(a)\cdot u}{t} \right|$$

$$\therefore \lim_{t \to 0} \frac{f(a+tu) - f(a)}{t} = Df(a)\cdot u$$

$$\| \quad D_u f(a)$$

(b) $D_u f(a) = \nabla f(a)\cdot u = \|\nabla f(a)\|\|u\| \cos\theta = \|\nabla f(a)\| \cos\theta.$

(c). $D_u f(a) = \|\nabla f(a)\| \cos\theta \leq \|\nabla f(a)\|$

& $D_{\frac{\nabla f(a)}{\|\nabla f(a)\|}} f(a) = \|\nabla f(a)\|$ since the angle between $\frac{\nabla f(a)}{\|\nabla f(a)\|}$ and $\nabla f(a)$ is 0.

#8. It's easy to show that $x \longmapsto (x, f(x))$ is differentiable in I because f is differentiable in I.

By chain rule, $F(x, f(x))$ is differentiable in I and

$$F_x(a,b) + F_y(a,b) f'(a) = DF(a,b) = 0$$

$\because F(x, f(x)) = 0 \quad \forall x \in I.$

$$\Rightarrow f'(a) = -\frac{F_x(a,b)}{F_y(a,b)}.$$