11.1 ③

ⓐ $f_x = \frac{4x^3(x^2+y^2) - 2x(x^4+y^4)}{(x^2+y^2)^2} = \frac{2x^5+4x^3y^2-2xy^4}{(x^2+y^2)^2}$ $y \neq 0$

$y=0 \Rightarrow f(x,0) = \begin{cases} x^2 & x \neq 0 \\ 0 & x=0 \end{cases}$, $f_x(x,0) = 2x$

$\lim_{y \to 0} f_x(x,y) = \frac{2x^5}{x^4} = 2x = f_x(x,0)$ $\therefore f_x$ is continuous on $\mathbb{R}^2$

ⓑ $y \neq 0$ $f(x,y) = (x^2-y^2)(x^2+y^2)^{-\frac{1}{3}}$

$f_x = 2x(x^2+y^2)^{-\frac{1}{3}} - \frac{1}{3}(x^2+y^2)^{-\frac{4}{3}} \cdot 2x(x^2-y^2)$

$= (x^2+y^2)^{-\frac{4}{3}}[2x(x^2+y^2) - \frac{2}{3}x(x^2-y^2)] = (x^2+y^2)^{-\frac{4}{3}}[\frac{4x^3}{3} + \frac{8}{3}xy^2]$

$y=0$, $f(x,0) = \begin{cases} x^{\frac{4}{3}} \\ 0 \end{cases}$ $\lim_{y \to 0} f_x(x,y) = x^{-\frac{8}{3}} \times \frac{4}{3}x^3 = \frac{4}{3}x^{\frac{1}{3}} = f_x(x,0)$

$f_x(x,0) = \frac{4}{3}x^{\frac{1}{3}}$

$\therefore f_x$ is continuous on $\mathbb{R}^2$

④

$f : [a,b] \times [c,d] \longrightarrow \mathbb{R}$ is continuous

$g : [a,b] \longrightarrow \mathbb{R}$ is integrable $\Rightarrow \exists M > 0$ s.t $|g(x)| \leq M \quad \forall x$

$\forall \varepsilon > 0$, $\forall y_1 \in [c,d]$ $\because f$ is conti , H is compact

$\therefore f$ is uniformly conti on H

$\exists \delta > 0$ as $d((x,y),(a,b)) < \delta \Rightarrow |f(x,y) - f(a,b)| < \frac{\varepsilon}{M(b-a)}$

as $d(y,y_1) \leq \delta$

$|F(y) - F(y_1)| = |\int_a^b g(x) f(x,y)dx - \int_a^b g(x) f(x,y_1)dx|$

$\leq \int_a^b |g(x)| \, |f(x,y) - f(x,y_1)| dx$

$\leq M \int_a^b \frac{\varepsilon}{M(b-a)} dx = \varepsilon$

$\therefore F(y)$ is conti 又 $[c,d]$ closed $\therefore F(y)$ is uniformly continuous

⑥ ⓐ $f(x)$ conti on $[0,1]$ $\therefore f(|x-1|)$ conti on $[0,2]$

令 $g(x,y) = e^{x^2y+y^2}$ is conti on $\mathbb{R}^2$

令 $F(y) = \int_0^2 \underbrace{f(|x-1|)g(x,y)}_{\text{conti}} dx$ ($\because F(y)$存在)

$\therefore$ By Thm 11.4 $\Rightarrow F(y)$ conti

$\therefore \lim_{y \to 0} F(y) = F(0) = \int_0^2 f(|x-1|)dx = \int_0^1 f(1-x)dx + \int_1^2 f(x-1)dx$

$= \int_0^1 f(y)dy + \int_0^1 f(z)dz$ $\begin{cases} y = 1-x \\ z = x-1 \end{cases}$

$= 2$ ok

ⓑ 令 $F(y) = \int_0^{\pi} \underbrace{f(x)\cos(y^5 + \sqrt[3]{y} + x)}_{\text{conti on } [0,\pi]\times[-1,1]}\,dx$

by Thm 11.4 $\Rightarrow$ $F(y)$ conti on $[-1,1]$

$\therefore \lim_{y\to 0} F(y) = F(0) = \int_0^{\pi} f(x)\cos x\,dx$

$= f(x)\sin x\big|_0^{\pi} - \int_0^{\pi} f'(x)\sin x\,dx$

$= -e$

$\therefore e + \lim_{y\to 0} F(y) = 0$ ※

7. ⓐ First $\forall y \in [0,1]$, $\frac{x\cos y}{\sqrt[3]{1-x+y}}$ is continuous, thus integrable (locally) on $(0,1)$

Second. let $M(x) = \frac{x}{\sqrt[3]{1-x}}$ then $|f(x,y)| \le M(x)$ on $(0,1)\times(0,1]$

and $\int_0^1 |M(x)|dx = \int_0^1 \frac{x}{\sqrt[3]{1-x}}dx = \int_0^1 \frac{1-u}{\sqrt[3]{u}}du$ which is integrable (absolutely)
$\left(= \frac{9}{10}\right)$

So by Thm 11.7 $\int_0^1 \frac{x\cos y}{\sqrt[3]{1-x+y}}dx$ converges uniformly

Thus by Thm 11.8 原式 $= \int_0^1 \lim_{y\to 0^+} \frac{x\cos y}{\sqrt[3]{1-x+y}}dx = \int_0^1 \frac{x}{\sqrt[3]{1-x}}dx = \frac{9}{10}$

ⓑ let $f(x,y) = \frac{e^{-xy}\sin x}{x}$ then:

(i) $\forall y \in [1,2]$ $\because |f(x,y)| < \frac{1}{x^2}$ and $\int_{\pi}^{\infty} \frac{1}{x^2}dx$ exists so $\int_{\pi}^{\infty} f(x,y)dx$ exists

(ii) $\because \forall y \in [1,2]$, $\int_{\pi}^{\infty} \left|\frac{\partial f}{\partial y}\right|dx \le \int_{\pi}^{\infty} e^{-xy}dx$ exists $\therefore \frac{\partial f}{\partial y}$ is locally integrable on $(\pi,\infty$

又 $\left|\frac{\partial f}{\partial y}\right| \le e^{-x}$ and $\int_{\pi}^{\infty} |e^{-xy}|dx$ exists

So 由 Thm 11.7 $\int_{\pi}^{\infty} \frac{\partial f}{\partial y}dx$ converges uniformly

So by Thm 11.9

原式 $= \int_{\pi}^{\infty} \frac{\partial f}{\partial y}dx\Big|_{y=1} = \int_{\pi}^{\infty} -e^{-x}\sin x\,dx = \frac{e^{-\pi}}{2}$

8. ⓐ $\left|\frac{\cos(x^2+y^3)}{\sqrt{x}}\right| \le \frac{1}{\sqrt{x}}$ $0 \le x \le 1$ $y \in \mathbb{R}$

又 $\int_0^1 \frac{1}{\sqrt{x}} = 2\sqrt{x}\big|_0^1 = 2 < \infty$ by Thm 11.7 原式 converges uniformly on $(-\infty,$

ⓑ $|e^{-xy}| \le e^{-x}$ as $\begin{matrix} 0 \le x < \infty \\ y \in (1,\infty) \end{matrix}$ $\int_0^{\infty} e^{-x}dx = -e^{-x}\big|_0^{\infty} = \frac{1}{e^0} = 1 < \infty$

by Thm 11.7 原式 converges uniformly on $[1,\infty)$

ⓒ $\int_0^{\infty} y e^{-xy}dx = -e^{-xy}\big|_0^{\infty} = -0-(-1) = 1$ 積分存在

$\forall a,b, a<b, [a,b] \subset (0,\infty)$, $y \in [a,b]$ $|ye^{-xy}| \le be^{-xa}$

$\int_0^{\infty} be^{-xa} = \frac{b}{a}e^{-xa}\big|_0^{\infty} = \frac{b}{a} < \infty$ by Thm 11.7 ... ok

if $y \in (0,\infty)$

$\int_{\alpha}^{\beta} ye^{-xy}dx = -e^{-xy}\big|_{\alpha}^{\beta} = \frac{1}{e^{\alpha y}} - \frac{1}{e^{\beta y}}$ 取 $\varepsilon = \frac{1}{e}$ $\forall \alpha, \beta$, $0 < \alpha < \beta < \infty$

$\exists y \in (0,\infty)$ $y = \frac{1}{\beta}$

$\left|\int_0^{\infty} ye^{-xy}dx - \int_{\alpha}^{\beta} ye^{-xy}dx\right| = \left|1 - \frac{1}{e^{\alpha y}} + \frac{1}{e^{\beta y}}\right| \ge \left|\frac{1}{e^{\beta y}}\right| = \frac{1}{e}$

$\therefore$ not converges uniformly