7. 令 $f(x)=1$ $\Rightarrow$ as $x\in E$ $f(x)>0$

as $x\notin E$ $f'(x)=0$.

10. ⓐ ($\Rightarrow$) 反証法

if $\exists r>0$ s.t. $E\cap B_r(a)\setminus\{a\}=\phi$

$\Rightarrow E\cap B_r(a)\subset\{a\}$

$\Rightarrow E\cap B_r(a)$ 頂多一点 (not infinitely many points)

$\Rightarrow$ a isn't a cluster point $\divideontimes$

($\Leftarrow$) $E\cap B_r(a)\setminus\{a\}\neq\phi,\ \forall r$ ...(*)

$\forall r_1>0 \overset{(*)}{\Rightarrow} E\cap B_{r_1}(a)\setminus\{a\}\neq\phi$

$\therefore \exists x_1\in E\cap B_{r_1}(a)\setminus\{a\}$, 且 $0<d(x_1,a)<r_1$

取 $0<r_2<d(x_1,a)<r_1$

$\overset{(*)}{\Rightarrow} E\cap B_{r_2}(a)\setminus\{a\}\neq\phi$

$\therefore \exists x_2\in E\cap B_{r_2}(a)\setminus\{a\}$

⋮ 依此做下去

取 $0<r_i<d(x_{i-1},a)<r_{i-1}<r_1$

$\overset{(*)}{\Rightarrow} E\cap B_{r_i}(a)\setminus\{a\}\neq\phi$

$\therefore \exists x_i\in E\cap B_{r_i}(a)\setminus\{a\}$

$\therefore$ 可得 $\{x_n\}$，且 $\underset{\forall i\in\mathbb{N}}{d(x_i,a)<r_1}$, $d(x_i,x_j)>0\ \forall i\neq j$

$\therefore a$ is a cluster point #

ⓑ $\forall E$ is a bounded infinite subset of $\mathbb{R}^n$

$\therefore \exists$ infinite distinct $\{x_n\}\in E \Rightarrow \{x_n\}$ is a bounded sequence in $\mathbb{R}^n$

by Thm 9.5, $\{x_n\}$ has a convergent subsequence $\{x_{k_n}\}$

$\{x_{k_n}\}\longrightarrow a$ i.e. $\forall r>0$ $E\cap B_r(a)\setminus\{a\}\neq\phi$

$\therefore$ a is a cluster point

(E has at least one cluster point)

2 ⓐ $\lim_{y\to0}\lim_{x\to0}\frac{\sin x\sin y}{x^2+y^2}=\lim_{y\to0}\frac{0}{y^2}=0$

$\lim_{x\to0}\lim_{y\to0}\frac{\sin x\sin y}{x^2+y^2}=\lim_{x\to0}\frac{0}{x^2}=0$

if $x=y \Rightarrow f(x,y)=\frac{\sin^2x}{2x^2}$ $\Rightarrow \lim_{x\to0}\frac{\sin^2x}{2x^2}=\frac{1}{2}\neq0$ $\therefore \lim_{(x,y)\to(0,0)}f(x,y)$ 不存在

ⓑ $\lim_{y\to0}\lim_{x\to0}\frac{x^2+y^4}{x^2+2y^4}=\lim_{y\to0}\frac{y^4}{2y^4}=\frac{1}{2}$

$\lim_{x\to0}\lim_{y\to0}\frac{x^2+y^4}{x^2+2y^4}=\lim_{x\to0}\frac{x^2}{x^2}=1\neq\frac{1}{2}$ $\therefore$ as $(x,y)\to(0,0)$ $f(x,y)$ 極限不存在

ⓒ $\lim_{y\to0}\lim_{x\to0}\frac{x-y}{(x^2+y^2)^\alpha}=\lim_{y\to0}\frac{-y}{(y^2)^\alpha}=0$ $(\because \alpha<\frac{1}{2})$

$\lim_{x\to0}\lim_{y\to0}f(x,y)=\lim_{x\to0}\frac{x}{(x^2)^\alpha}=0$ $(\because \alpha<\frac{1}{2})$

令 $x=r\cos\theta$, $y=r\sin\theta$ 代入 $\Rightarrow \lim_{r\to0}\frac{r(\cos\theta-\sin\theta)}{r^{2\alpha}}$

$=\lim_{r\to0}r^{1-2\alpha}\underbrace{(\cos\theta-\sin\theta)}_{\text{bounded}}=0$

$(\lim_{r\to0}r^{1-2\alpha}=0)$

$\therefore f(x,y)$ 極限存在且 $=0$ as $(x,y)\to(0,0)$

3 ⓐ 令 $x=r\cos\theta$, $y=r\sin\theta$ $\lim_{(x,y)\to(0,0)}f(x,y)=\lim_{r\to0}\frac{r^3(\cos^3\theta-\sin^3\theta)}{r^2}$

$=\lim_{r\to0}r(\cos^3\theta-\sin^3\theta)=0$ 極限存在

ⓑ $f(x,y)=\frac{|x|^\alpha y^4}{x^2+y^4}\leq|x|^\alpha$ $\forall (x,y)\neq(0,0)$

$\therefore \limsup_{(x,y)\to(0,0)}f(x,y)\leq\limsup_{(x,y)\to(0,0)}|x|^\alpha=0$

$\because f(x,y)\geq0$

$\therefore \liminf_{(x,y)\to(0,0)}f(x,y)\geq0$

$\because \limsup_{(x,y)\to(0,0)}f(x,y)=\liminf_{(x,y)\to(0,0)}f(x,y)=0$

$\therefore \lim_{(x,y)\to(0,0)}f(x,y)=0$ #

9-3

2 ⓐ $f((0,1))=(0,1)$ $\quad g((0,1))=(1,\infty)$

$f([0,1))=[0,1)$ $\quad g([0,1))=(1,\infty)\cup\{0\}$

$f([0,1])=[0,1]$ $\quad g([0,1])=[1,\infty)\cup\{0\}$

ⓑ $f^{-1}((-1,1))=[0,1)$ $\quad g^{-1}((-1,1))=(1,\infty)\cup(-\infty,-1)$

$f^{-1}([-1,1])=[0,1]$ $\quad g^{-1}([-1,1])=[1,\infty)\cup(-\infty,-1]$

3.

ⓐ ∵ g is continuous at b

∴ $\forall \varepsilon>0\ \exists \delta>0\quad d(y,b)<\delta \Rightarrow d(g(y),g(b))<\varepsilon$

∵ $b:=\lim\limits_{x\to a} f(x)$ ∴ $\exists \delta_1>0,\ d(x,a)<\delta_1$ s.t. $d(f(x),b)<\delta$

∴ $d(g(f(x)),g(b))<\varepsilon$

∴ $\lim\limits_{x\to a} g\circ f(x)=g(b)$ #

ⓑ 証法如同ⓐ (f is continuous at a $\Rightarrow f(a)=b=\lim\limits_{x\to a}f(x)$)

4. 只須 consider $\lim\limits_{\substack{x\to y\\ x\neq y}} f(x,y)=\lim\limits_{\substack{x\to y\\ x\neq y}} e^{-\frac{1}{|x-y|}}=0=f(x,x)$

∴ $f(x,y)$ is continuous on $\mathbb{R}^2$ #

5. $\forall$ closed subset E of $\mathbb{R}^m$

(a)⇒(b)

if $f^{-1}(E)=\phi$ is closed in $\mathbb{R}^n$

if $f^{-1}(E)\neq\phi$

consider x is $f^{-1}(E)$ 的 limit point

$\Rightarrow \exists \{x_n\}\in f^{-1}(E)\subset B$, $x_n\longrightarrow x$ as $n\to\infty$

∵ B is closed in $\mathbb{R}^n$

x is a limit point of B ∴ $x\in B$

∵ f continuous on B

∴ $\lim\limits_{n\to\infty} f(x_n)=f(x)$ i.e. $f(x)$ is E 的 limit point

∵ E is closed in $\mathbb{R}^m$ ∴ $f(x)\in E$ i.e. $x\in f^{-1}(E)$

∴ $f^{-1}(E)$ is closed on $\mathbb{R}^n$

(b) $\Rightarrow$ (a) 反証法

if $x \in B$ 且 f 在 x 點上不連續

$\exists\, \varepsilon > 0$, $\exists\, x_n \in B$, $x_n \to x$ 且 $|f(x_n) - f(x)| \geq \varepsilon$

取 $E = \mathbb{R}^n \setminus B_\varepsilon(f(x))$ is closed

$\therefore \{x_n\} \in f^{-1}(E)$, $x \notin f^{-1}(E)$

but $f^{-1}(E)$ is closed, x is a limit point of $f^{-1}(E)$

$\therefore x \in f^{-1}(E)$ $\rightarrow\leftarrow$ #

7. ⓐ $\because$ H is a closed and bounded subset of $\mathbb{R}^n$

$\therefore$ H is compact

$\because$ f is continuous, H is compact

$\therefore$ $f(H)$ is compact

令 $g(x) = \|x\|$, $g: \mathbb{R}^m \longrightarrow \mathbb{R}$, g is continuous. on $\mathbb{R}^m$

By 3.(b) $g \circ f(x) = \|f(x)\|$ is continuous on H.

by Extreme Value Theorem $\|f\|_H = \sup_{x \in H} \|f(x)\|$ is finite and

$\exists\, x_0 \in H$ s.t. $\|f(x_0)\| = \|f\|_H$

ⓑ ($\Rightarrow$) $\|f_k - f\|_H \to 0$ as $k \to \infty$

$\forall\, \varepsilon > 0$ $\exists\, N > 0$, $k \geq N$ $\|f_k - f\|_H < \varepsilon$

$\forall\, x$ $\|f_k(x) - f(x)\| \leq \|f_k - f\|_H < \varepsilon$ $\therefore$ $f_k \to f$ uniformly converge on H.

($\Leftarrow$) $\forall\, \varepsilon > 0$ $\exists\, N \in \mathbb{N}$, $k \geq N$ and $\forall x \in H$

s.t. $\|f_k(x) - f(x)\| < \varepsilon$

by a $\exists\, x_k$ s.t. $\|f_k(x_k) - f(x_k)\| = \|f_k - f\|_H$

$\therefore \|f_k - f\|_H = \|f_k(x_k) - f(x_k)\| < \varepsilon$

$\therefore \|f_k - f\|_H \to 0$ as $k \to \infty$

ⓒ ($\Rightarrow$) $\forall\, \varepsilon > 0$ $\exists\, N \in \mathbb{N}$, $\forall\, k, j \geq N$ s.t. $\|f_k(x) - f(x)\| < \frac{\varepsilon}{3}$
$\forall\, x \in H$ $\|f_j(x) - f(x)\| < \frac{\varepsilon}{3}$

$\|f_k(x) - f_j(x)\| \leq \|f_k(x) - f(x)\| + \|f(x) - f_j(x)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2}{3}\varepsilon$

取 sup $\|f_k - f_j\|_H \leq \frac{2}{3}\varepsilon < \varepsilon$

⑤ $(\Leftarrow)$ $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$, $\forall k, j \geq N$ s.t. $\|f_k - f_j\|_H < \varepsilon$

$\therefore$ fix x $|f_k(x) - f_j(x)| \leq \|f_k - f_j\|_H < \varepsilon$

$\Rightarrow \{f_k(x)\}$ is cauchy sequence in $\mathbb{R}^m$

$\therefore \exists f(x)$ s.t. $f_k(x) \to f(x)$ pointwise converge

$\forall x \in H$, $\|f_k(x) - f_j(x)\| \leq \|f_k - f_j\| < \varepsilon$

as $j \to \infty$ $\|f_k(x) - f(x)\| \leq \varepsilon$

$\therefore$ $f_k(x) \to f(x)$ uniformly converge.

8、 if $x \in E \setminus D$, $\because \bar{D} = E$ $\therefore \exists \{x_p\} \in D$ 且 $x_p \to x$ in $\mathbb{R}^n$

$\therefore \{x_p\}$ is Cauchy sequence

$\because f$ is uniformly continuous on D

$\therefore \forall \varepsilon > 0$ $\exists \delta > 0$ as $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

$\because \{x_p\}$ is cauchy sequence

$\therefore$ N 夠大 $k, l > N$ $|x_k - x_l| < \delta \Rightarrow |f(x_k) - f(x_l)| < \varepsilon$

$\therefore \{f(x_p)\}$ is a cauchy sequence in $\mathbb{R}^m$

$\therefore \lim_{p \to \infty} f(x_p)$ 存在

$\forall \{x_p\} \to x$ N 夠大 $p \geq N$ s.t. $|x_p - x| < \delta$

$\{x_p'\} \to x$ N' 夠大 $p' \geq N'$ s.t. $|x_p' - x| < \delta$

取 $\max\{N, N'\} = N_2$

as $P_1, P_2 > N_2$ $|f(x_{p_1}) - f(x'_{p_2})| \leq |f(x_{p_1}) - f(x)| + |f(x) - f(x'_{p_2})|$

$< \varepsilon + \varepsilon = 2\varepsilon$

$\therefore \lim_{p \to \infty} f(x_p) = \lim_{p \to \infty} (x_p')$ $\therefore$ $g(x) \equiv \begin{cases} f(x) & x \in D \\ \lim_{n \to \infty} f(x_n) & \text{if } a \in E \setminus D \text{ 且 } x_n \to a \end{cases}$

$g(x)$ is a well-defined

$\forall \varepsilon > 0$, $\because f$ is uniformly continuous on D $\therefore \exists \delta > 0$, $x, y \in D$ $|x - y| < \delta$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{3}$

consider $\forall |x - y| < \delta$, $x, y \in \bar{D}$

Case 1. $x, y \in D \Rightarrow |g(x) - g(y)| = |f(x) - f(y)| < \frac{\varepsilon}{3} < \varepsilon$ ok

case 2. $x \in D$, $y \in \bar{D} \setminus D \Rightarrow \exists \{x_n\} \in D$, $x_n \to y$ as $n \to \infty$

$\exists N \in \mathbb{N}$, $\forall n \geq N$ s.t. $|x_n - y| < \delta - |x - y|$

$\Rightarrow |x_n - x| \leq |x_n - y| + |y - x| < (\delta - |x - y|) + |y - x| = \delta$

$\therefore |f(x_n) - f(x)| < \varepsilon/3$ $\forall n \geq N$ $\because \lim_{n \to \infty} f(x_n) = g(y)$ (by def)

$\therefore$ as $n \to \infty$ $\Rightarrow |g(y) - f(x)| \leq \frac{\varepsilon}{3} < \varepsilon$ ($\because g(x) = f(x)$ $x \in D$)

$\therefore |g(y)-g(x)| < \varepsilon$

case 3. $x, y \in \overline{D} \setminus D$, $|x-y| < \delta$

$\Rightarrow \exists \{x_n\} \in D \quad x_n \to y$ as $n \to \infty$.

$\exists N \in \mathbb{N} \quad \forall n \geq N$ s.t $|x_n - y| < \delta - |x-y|$

$\Rightarrow |x_n - x| \leq |x_n - y| + |y - x| < \delta$

由 case 2 知 $|g(x_n) - g(x)| \leq \frac{\varepsilon}{3}$

令 $n \to \infty$ $|g(y) - g(x)| \leq \frac{\varepsilon}{3} < \varepsilon$

$\left(\because g(y) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) \right)$

$\therefore g$ is uniformly continuous on $\overline{D}$

$\therefore g$ is continuous on $\overline{D}$ #