Differential Geometry Homework #2 due 3/22

1. Let $\omega \in \mathfrak{X}^*(M)$ and γ a smooth curve on M. Assume that $F : M \to N$ is smooth. Show that

$$\int_\gamma F^*\omega = \int_{F\circ\gamma} \omega.$$

2. Show that the restriction of $\omega = x^1 dx^2 - x^2 dx^1 + x^3 dx^4 - x^4 dx^3$ of $\mathbb{R}^4$ to the sphere $\mathbb{S}^3$ is never zero on $\mathbb{S}^3$. [Boothby, p.183 #6]

3. Show that TM is trivial if and only if T^*M is trivial.

4. Let σ be a covariant k-tensor field on M. Let (x^i) and $(\tilde{x}^i)$ be two smooth local coordinates. In local coordinates, we write

$$\sigma = \sigma_{i_1\cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} = \tilde{\sigma}_{i_1\cdots i_k} d\tilde{x}^{i_1} \otimes \cdots \otimes d\tilde{x}^{i_k}.$$

Please write the transformation law between $\sigma_{i_1\cdots i_k}$ and $\tilde{\sigma}_{i_1\cdots i_k}$.

5. Show that $\Phi(A, B) = \mathrm{tr}\ {}^tAB$, the trace of the transpose of A times B, defines a symmetric bilinear form on $\mathbb{R}^{n\times n}$. Is it positive-definite? [Boothby, p.187 #9]