Differential Geometry Homework #1 due 3/15

1. Let $M = \mathbb{R}^2$ and $X = x^2 \frac{\partial}{\partial x}$. Find the flow domain W and the flow $\theta : W \to M$.

2. Let $M = GL(2, \mathbb{R})$ and define an action of $\mathbb{R}$ on M by the formula

$$\theta(t, A) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot A, \quad A \in GL(2, \mathbb{R})$$

with the dot denoting matrix multiplication. Find the infinitesimal generator. [Boothby, p.130 #7]

3. Let X be a smooth vector field on M and let $F : I(p) \to M$ be the integral curve determined by $F(0) = p$. Suppose for some real number $c > 0$, $F(c) = F(0)$. Show that this implies $I(p) = \mathbb{R}$ and $F(t) = F(t + c)$ for all $t \in \mathbb{R}$. If $X_p \neq 0$, then prove that there is a diffeomorphism $G : \mathbb{S}^1 \to M$ and a number c_0, $0 < c_0 \leq c$, such that $F = G \circ \pi$, with $\pi : \mathbb{R} \to \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ denoting the mapping $\pi(t) = e^{2\pi i(t/c_0)}$. [Boothby, p.146 #3]

4. Given $p \in M$, show that if $I(p)$ is bounded for a C^∞-vector field X on M, then $t \to \theta(t, p)$ is an imbedding of $I(p)$ in M. [Boothby, p.146 #4]

5. Show that for any Lie group G the mapping $\mu : G \to G$ defined by $\mu(g) = g^{-1}$ takes left-invariant vector fields to right invariant vector fields. [Boothby, p.151 #5]

Note: here $I(p)$ is $W^{(p)}$ we defined in class.