Differential Geometry Homework #9 due 12/14

1. Let e be the identity of the Lie group G and V be any neighborhood of e. Then there exists a neighborhood W of e such that $W^{-1}W \subset V$, and a neighborhood U of e such that $U^2 = UU = \{g_1 g_2 : g_1, g_2 \in U\} \subset V$.

2. (Boothby, p.95, #3) Let G act transitively on X and let x_0 be a point of X. Define $\tilde{F} : G \to X$ by $\tilde{F}(g) = gx_0$. Prove: (i) that there is a unique one-to-one mapping $F : G/G_{x_0} \to X$ such that $\tilde{F} = F \circ \pi$, $\pi : G \to G/G_{x_0}$, the natural projection to cosets, (ii) that $\tilde{F}$ and F are continuous if the action is continuous and in this case F is a homomorphism if and only if $\tilde{F}$ is open.

3. (Boothby, p.95, #10) Let G be a Lie group and H be a closed subgroup and define a left action $\theta : G \times G/H \to G/H$ by $\theta(g, xH) = (gx)H$. Show that this action of G on the coset space G/H is continuous and that the isotropy group of $[e] = H$ is exactly H itself.

4. Let $\mathrm{SO}(n)$ be the special orthogonal group, i.e., $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R})$. Show that when $n > 1$, $\mathrm{SO}(n)$ acts transitively on $\mathbb{S}^{n-1}$.