Differential Geometry Homework #8 due 12/7

1. (Boothby, p.81,#1) Let $F : N \to M$ be a one-to-one immersion which is proper, i.e., the inverse image of any compact set is compact. Show that F is an imbedding and that its image is closed regular submanifold of M.

2. Let $\iota : N \to M$ be a one-to-one immersion, X be a manifold, and $f : X \to M$ be a smooth map with $f(X) \subseteq \iota(N)$.
(a) Show by an example that $\iota^{-1} \circ f : X \to N$ may fail to be continuous.
(b) If $\iota^{-1} \circ f$ is continuous, prove that it is smooth.

3. Let $\mathfrak{M}(m, n)$ be the set of real $m \times n$ matrices and $\mathfrak{M}(m, n; k)$ be the set of all $m \times n$ matrices of rank k. This exercise is to show that $\mathfrak{M}(m, n; k)$ is a regular submanifold of $\mathfrak{M}(m, n)$ of dimension $k(m + n - k)$.
(a) For every $M_0 \in \mathfrak{M}(m, n; k)$ there exist permutation matrices P and Q such that

$$PM_0Q = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix},$$

where A_0 is $k \times k$ non-singular matrix.
(b) There is some $\varepsilon > 0$ such that A is non-singular whenever all entries of $A - A_0$ are $< \varepsilon$.
(c) If

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where the entries of $A - A_0$ are $< \varepsilon$, then X has rank k if and only if $D = CA^{-1}B$.
(d) $\mathfrak{M}(m, n; k)$ is a regular submanifold of $\mathfrak{M}(m, n)$ of dimension $k(m+n-k)$ for all $k \leq m, n$.