Differential Geometry Homework #7 due 11/30

1. We define a k-frame in $\mathbb{R}^n$ to be a linear independent set $\mathbf{x}$ of k elements of $\mathbb{R}^n$:

$$x_1 = (x_1^1, \cdots, x_1^n)$$
$$\vdots$$
$$x_k = (x_k^1, \cdots, x_k^n).$$

A k-frame in $\mathbb{R}^n$ may be identified with the $k \times n$ matrix, which we also denoted by $\mathbf{x}$, whose rows are $x_1, \cdots, x_k$. Let $F(k,n)$ be the set of k-frame in $\mathbb{R}^n$.
(a) Show that $F(k,n)$ is differentiable manifold of dimension kn.
(b) Let $\mathbf{x}$ and $\mathbf{y}$ be two k-frames. We define an equivalence relation $\sim$ by saying that

$$\mathbf{x} \sim \mathbf{y} \quad \text{if} \quad \mathbf{y} = \mathbf{a}\mathbf{x}, \quad \mathbf{a} \in GL(k, \mathbb{R}).$$

Then $G(k,n)$ is identified with $F(k,n)/\sim$. Let $\pi : F(k,n) \to G(k,n)$ be the quotient map. Show that π is in fact C^∞.

2. (Boothby p.74 #3) On the "figure eight" image $\tilde{N}$ of $\mathbb{R}$ on page 72. Let the topology and C^∞ structure be given by the one-to-one immersion G (please refer to page 72 for the definition of G). Show that reflection $H : (x^1, x^2) \to (x^1, -x^2)$ of $\mathbb{R}^2$ in the x^1 axis, although it maps $\tilde{N}$ onto itself, is not a diffeomorphism of $\tilde{N}$.

3. Let M be a compact manifold of dimension n and $F : M \to \mathbb{R}^n$ be a smooth map. Prove that F cannot everywhere be non-singular.

4. Consider S^1 as the unit circle in the complex plane. Define a mapping $F : \mathbb{R} \to \mathbb{S}^1 \times \mathbb{S}^1$ by setting $F(t) = (e^{2\pi i t}, e^{2\pi i \alpha t})$, where α is an irrational number. Prove that F is an injective immersion with dense range.

5. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\Phi(x, y) = x^3 + xy + y^3 + 1.$$

Which level sets of Φ are regular submanifolds of $\mathbb{R}^2$.