Differential Geometry Homework #5 due 11/9

1. Let $X = f_1(x_1, x_2)\frac{\partial}{\partial x_1} + f_2(x_1, x_2)\frac{\partial}{\partial x_2}$ be a smooth vector field on $\mathbb{R}^2$.
(a) If $\alpha(t) = (x_1(t), x_2(t))$ is an integral curve for X, what differential equation do $x_1(t)$ and $x_2(t)$ satisfy?
(b) Suppose that $\alpha(t)$ and $\beta(t)$ are two integral curves for X and $\alpha(0) = \beta(1)$. What can you say about the relation between α and β at other values of t?
(c) If $f_1(x_1, x_2) = x_2$ and $f_2(x_1, x_2) = -x_1$, describe the integral curves for X.

2. Let $X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$, $Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$, and $Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.
(a) Determine the flows generated by X, Y, and Z.
(b) Let θ and ϕ denote the flows of X and Y respectively. Show that these flows do not commute by finding explicit times s, t such that $\theta_s \circ \phi_t \neq \phi_t \circ \theta_s$.

3. Given $A \in \mathfrak{M}(n)$, view the right translation operation

$$R_A : GL(n) \to \mathfrak{M}(n)$$

as a vector field $R_A \in \mathfrak{X}(GL(n))$.
(a) For

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

show that the local flow generated by R_A has the formula

$$\Phi_t(Q) = Q \cdot \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},$$

for all $t \in \mathbb{R}$ and $Q \in GL(2)$.
(b) Note that the formal definition

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^n}{n!}A^n + \cdots$$

yields

$$e^{tA} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Compute e^{tB} for the matrix

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and make an educated guess of a flow on $GL(2)$ generated by R_B. Prove that your guess is correct. (Conlon's exercise)

4. Consider the map $f : \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$(u, v) = f(w, x, y, z) = (w^2 + x, w^2 + x^2 + y^2 + z^2 + x).$$

Prove that $(0, 1)$ is a regular value of f and the preimage $M = f^{-1}(0, 1)$ is homeomorphic to the standard 2-sphere $\mathbb{S}^2$ in $\mathbb{R}^3$.