Differential Geometry Homework #3 due 10/26

1. Let U be an open set of $\mathbb{R}^n$ and $p \in U$. Show that the tangent space $T_p(U)$ of U at p is a vector space of dimension n.

2. Let $p \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ and define

$$T_p(\mathbb{S}^n) = \{\langle s \rangle_p \in T_p(\mathbb{R}^{n+1}) | s : (-\epsilon, \epsilon) \to \mathbb{R}^{n+1} \text{ has image}(s) \subset S^n\}.$$

Prove that $T_p(\mathbb{S}^n)$ is the linear subspace of $\mathbb{R}^{n+1} = T_p(\mathbb{R}^{n+1})$ consisting of all $v \perp p$.

3. Let $\mathcal{G}_p^k$ be the germs of $C^k(U, p)$ functions. Denote $\mathcal{G}_g^\infty = \mathcal{G}_p$. Is the identification $T(\mathcal{G}_p^k) \cong T_p(U)$ still valid when $k < \infty$?

4. Let $\mathcal{G}_p^* \subset \mathcal{G}_p$ be the kernel of the evaluation map e_p and let $\mathcal{G}_p^{**} \subset \mathcal{G}_p^*$ be the vector subspace spanned by the germs of functions gf, where $g, f \in C^\infty(U, p)$ and $f(p) = g(p) = 0$. Prove that the quotient space $\mathcal{G}_p^*/\mathcal{G}_p^{**}$ is canonically isomorphic to the vector space dual of $T_p(U)$. The quotient space $\mathcal{G}_p^*/\mathcal{G}_p^{**}$ is defined in the sense that for $f_1, f_2 \in \mathcal{G}_p^*$, $f_1 \sim f_2$ whenever $f_1 - f_2 \in \mathcal{G}_p^{**}$.