Differential Geometry Homework #11 due 1/4

**1**. (Boothby, p.122 #11) Let $F : \tilde{M} \to M$ be a $C^\infty$ covering and $Y$ any $C^\infty$ vector field on $M$. Show that there is a unique $C^\infty$ vector field $X$ on $\tilde{M}$ such that $X$ and $Y$ are $F$-related.

**2**. (Boothby, p.122 #12) Show that any $C^\infty$ vector field $Y$ on $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is the restriction of a $C^\infty$ vector field $X$ on $\mathbb{R}^n$.

**3**. We have shown that if $F : M \to N$ is a diffeomorphism then for every $X \in \mathfrak{X}(M)$ there is a unique $Y \in \mathfrak{X}(N)$ which is $F$-related to $X$. Does the same result hold if one replaces that $F$ is a diffeomorphism by that $F$ is smooth and bijective?

**4**. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras and $F : \mathfrak{g} \to \mathfrak{h}$ a linear map. $F$ is called a Lie algebra homomorphism if it preserves Lie brackets, i.e., $F[X, Y] = [FX, FY]$. Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are called isomorphic if there exists an invertible Lie algebra homomorphism from $\mathfrak{g}$ onto $\mathfrak{h}$. Now let $A \subset \mathfrak{X}(\mathbb{R}^3)$ be a subspace with the base:

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\ Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Show that $A$ is a Lie subalgebra of $\mathfrak{X}(\mathbb{R}^3)$, which is isomorphic to $\mathbb{R}^3$ with the cross product.

**5**. Use the stereographic projection to show that there exists a smooth vector field on $\mathbb{S}^2$ which vanishes at exactly one point.