Differential Geometry Homework #1 due 10/05

1. Let X be the set $\mathbb{R}\times\{0\}\cup\mathbb{R}\times\{1\}$ considered as a subset of $\mathbb{R}^2$. Define the equivalence relation $\sim$ by identifying $(x,0)$ and $(x,1)$ whenever $x \neq 0$. Show that $X/\sim$ with quotient topology is second countable and locally Euclidean, but not Hausdorff.

2. Use stereographic projection (not the method we used in class) to show that $\mathbf{S}^n$ is a topological manifold of dimension n.

3. Let $X_i,\ i = 1, \cdots, n$, be topological spaces. Topologize $X = X_1 \times \cdots \times X_n$ by the product topology. Show that any projection map $\pi_i : X \mapsto X_i$ is a quotient map.

4. Let $\pi : X \mapsto Y$ be a quotient map and U be a saturated open set of X. Show that $\pi|_U$ is a quotient map. Also, show by an example that "saturated" is necessary in the proceeding result.

5. Consider the following subset of $\mathbb{R}^2$: $X = A_+ \cup A_- \cup B$ with

$$A_+ = \{(x,y) : x \geq 0, y = +1\},$$

$$A_- = \{(x,y) : x \geq 0, y = -1\},$$

and

$$B = \{(x,y) : x < 0, y = 0\}.$$

Define a topology as follows: We use the subspace topology (open intervals as a basis) on $A_+ \setminus \{(0,1)\}$, $A_- \setminus \{(0,-1)\}$, and B; then for $\varepsilon > 0$ we let $N_\varepsilon^\pm = \{(x,\pm 1) : 0 \leq x < \varepsilon\} \cup \{(x,0) : -\varepsilon \leq x < 0\}$ and use N_ε^+ and N_ε^- as a basis of neighborhoods of $(0,1)$ and $(0,-1)$, respectively. Show that the space X is locally Euclidean but is not a topological manifold.