Solution to 94 Midterm # 3

$f''(x) > 0$ in $[-1, 5]$ implies that $f'(x)$ is a strictly *increasing* function in $(-1, 5)$. Using $f(0) = 2$, $f(1) = 0$ and the *Mean Value Theorem*, there exists $0 < c_1 < 1$ such that

$$f'(c_1) = \frac{f(1) - f(0)}{1 - 0} = 0 - 2 = -2 < 0.$$

On the other hand, $f(2) = 3$. So *MVT* again implies that there exists $1 < c_2 < 2$ such that

$$f'(c_2) = \frac{f(2) - f(1)}{2 - 1} = 3 - 0 = 3 > 0.$$

Hence we must have $0 < c_1 < c_0 < c_2 < 2$ with $f'(c_0) = 0$ and $f'(x) < 0$ if $x < c_0$ and $f'(x) > 0$ if $x > c_0$. In other words, $f(c_0)$ is a minimum value. So (d) is true. Also, since $f'(x)$ is increasing in $(-1, 5)$ and $f'(c_1) < 0$ with some $c_1 > 0$, we have $f'(0) < f'(c_1) < 0$. So (a) is false.

Now $f(3) = 7$, from *MVT* we get

$$f'(c_3) = \frac{f(3) - f(2)}{3 - 2} = 7 - 3 = 4 \quad \text{for some} \quad c_3 \in (2, 3).$$

Therefore, we have that $c_2 < 2 < c_3$ and $f'(c_2) = 3$, $f'(c_3) = 4$. Since $f'(x)$ is increasing, $3 < f'(2) < 4$. Hence (c) holds.

Finally, we know that $f'(2) > 3 > 0$ and $f'(x)$ is increasing. So $f'(x) > 0$ at least in $(2, 5)$. Therefore, f itself is strictly increasing in $(2, 5)$. Now $f(3) = 7$ and $3 < 4$ implies $f(4) > 7$. Thus (b) is true.

In summary, the correct answer is (b), (c), (d).