

# A NONVANISHING THEOREM FOR $\mathbf{Q}$ -DIVISORS

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ABSTRACT. We prove a non-vanishing theorem of the cohomology  $H^0$  of the adjoint divisor  $K_X + \lceil L \rceil$  where  $\lceil L \rceil$  is the round up of a nef and big  $\mathbf{Q}$ -divisor  $L$ .

## 1. INTRODUCTION

We work over the complex numbers field  $\mathbf{C}$ . The motivation of this note is to find an effective version of the famous non-vanishing theorem of Kawamata and Shokurov (see [KMM], [Sh]). We propose the following:

**Conjecture 1.1.** *Let  $X$  be a nonsingular projective variety. Let  $L$  be a  $\mathbf{Q}$ -divisor on  $X$  satisfying the conditions below:*

- (1)  $L$  is nef and big,
- (2)  $K_X + L$  is nef, and
- (3) either  $L$  is a Cartier integral divisor, or  $L$  is effective.

*Then  $H^0(X, K_X + \lceil L \rceil) \neq 0$ , where  $\lceil L \rceil$  is the round up of  $L$ .*

This kind of non-vanishing problem has been considered by Ambro [Am], A. Chen-Hacon [CH], Kawamata [Ka], Kollar [Ko], Takayama [Ta], and others. When  $L$  is an integral Cartier divisor, Kawamata [Ka] has proved the above Conjecture 1.1 if either  $\dim X = 2$ , or  $\dim X = 3$  and  $X$  is minimal (i.e., the canonical divisor  $K_X$  is nef).

Conjecture 1.1 is slightly different from that of Kawamata's in [Ka]. It is somewhat general in the sense that the divisor  $L$  in question is not assumed to have integral coefficients. It is precisely this non-Cartierness of  $L$  that causes a lot of trouble when estimating  $h^0(X, K_X + \lceil L \rceil)$ . To elaborate, the Kawamata-Viehweg vanishing ([KV], [Vi1]) implies that  $h^0(X, K_X + \lceil L \rceil) = \chi(K_X + \lceil L \rceil)$  when the fractional part of  $L$  is of normal crossings. However, the formula for  $\chi$  may not be effective because  $\lceil L \rceil$  may not be nef and hence  $\lceil L \rceil \cdot (K_X + \lceil L \rceil)$  may not be non-negative. The worse thing is that as remarked in a recent paper of [Xi], there are  $\mathbf{Q}$ -Fano 2-folds and 3-folds (see [Fl]) with vanishing  $H^0(X, K_X + (-2K_X))$ .

Despite of the observations above, in [Xi] it is proved that  $H^0(X, (D - K_X) + K_X) \neq 0$  for Picard number one Gorenstein del Pezzo surface  $X$  and nef and big  $\mathbf{Q}$ -Cartier Weil divisor  $D$ . In this note we shall prove the following which is a consequence of Theorems 4.1, 5.1, 8.1 and 8.2 (for the case of integral Cartier  $L$ , see [Ka]).

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**Theorem 1.2.** *Let  $X$  be a nonsingular projective surface. Suppose that  $X$  and  $L$  satisfy the conditions (1) - (3) in Conjecture 1.1. Suppose further that  $X$  is relatively minimal. Then either  $H^0(X, K_X + \lceil L \rceil) \neq 0$  or  $H^0(X, K_X + 4L_{\text{red}}) \neq 0$ .*

The second conclusion may occur when  $K_X$  is nef (and the Kodaira dimension  $\kappa(X) \geq 1$ ). In this case, the conditions in Conjecture 1.1 are automatically satisfied whenever  $L$  is nef and big. So the coefficients of  $L$  can be as small as one likes. Thus the non-vanishing of  $H^0(X, K_X + \lceil L \rceil)$  is equivalent to that of  $H^0(X, K_X + L_{\text{red}})$ , which is stronger than our conclusion. Remark 8.4 shows that it is hard to replace the coefficient "4" in the theorem above by "1".

In Sections 3 and 6 (Theorems 3.1 and Theorem 6.1), we prove the following non-vanishing results without assuming the condition (3) in Conjecture 1.1, and the proof presented for the first assertion is applicable to higher dimensional varieties.

**Theorem 1.3.** *Let  $X$  and  $L$  be as in Conjecture 1.1 satisfying the first two conditions only. Then  $H^0(X, K_X + \lceil L \rceil) \neq 0$  if either*

- (i)  *$X$  is a surface with irregularity  $q(X) > 0$ , or*
- (ii)  *$X$  is a relatively minimal elliptic surface with  $\kappa(X) = -\infty$  and  $K_X + L$  nef and big.*

*Remark 1.4.* (1) In Example 2.6, we construct an example of pair  $(X, L)$  satisfying the conditions (1) and (2) in Conjecture 1.1 (indeed, both  $L$  and  $K_X + L$  are nef and big) but with  $H^0(X, K_X + \lceil L \rceil) = 0$ . So an extra condition such as the (3) in Conjecture 1.1 is necessary.

(2) The same example shows that in Kollar's result [Ko] on non-vanishing of  $H^0(X, K_X + M)$  for big divisor  $M$ , the "bigness" assumption on the fundamental group  $\pi_1(X)$  is necessary, because in (1) the  $M := \lceil L \rceil \geq L$  is big and  $\pi_1(X) = (1)$ .

(3) The example also shows the necessity to assume the nefness of the Cartier integral divisor  $D$  (with  $(X, B)$  klt and  $D - (K_X + B)$  nef and big) in Kawamata's conjecture [Ka] for the non-vanishing of  $H^0(X, D)$ . Indeed, in the example, we have  $\lceil L \rceil = L + B$  with  $B$  a simple normal crossing effective divisor so that  $[B] = 0$ , whence  $(X, B)$  is klt. To be precise, let  $D := \lceil L \rceil$ . Then  $D - (K_X + B) = \lceil L \rceil - B = L$  is nef and big,  $D = K_X + \lceil L \rceil$ , and  $D$  is not nef for  $D \cdot D_i = -1$  with the notation in the example.

We end the Introduction with:

*Remark 1.5.* Consider a fibred space  $f : V \rightarrow C$  where  $V$  is a nonsingular projective variety and  $C$  a complete curve. Assume  $L$  is a nef and big normal crossing  $\mathbf{Q}$ -divisor such that  $K_V + L$  is nef. The well-known positivity says that  $f_*(\omega_{V/C} \otimes \mathcal{O}_V(\lceil L \rceil))$  is positive whenever it is not equal to 0. Pick up a general fibre  $F$  of  $f$ . The induction of the non-vanishing problem on  $F$  may imply that

$$\text{rk}(f_*(\omega_{V/B} \otimes \mathcal{O}_V(\lceil L \rceil))) = h^0(F, K_F + \lceil L \rceil|_F) \geq h^0(F, K_F + \lceil L \rceil|_F) \neq 0.$$

The positivity of  $f_*(\omega_{V/B} \otimes \mathcal{O}_V(\lceil L \rceil))$  has direct applications in studying properties of the moduli schemes for polarized manifolds. Please refer to [Vi2] for more details.

The above remark shows one aspect of the importance of the effective non-vanishing problem.

## 2. SOME PREPARATIONS AND AN EXAMPLE

We begin with:

**Definition 2.1.** A reduced connected divisor  $\Gamma$ , with only simple normal crossings, is a rational tree if every component of  $\Gamma$  is a rational curve and the dual graph of  $\Gamma$  is a tree (i.e., it contains no loops).

Before proving Proposition 2.4 below, we need two lemmas in advance.

**Lemma 2.2.** *Let  $D = \sum_{j=1}^n D_j$  be a reduced connected divisor on a nonsingular projective surface  $X$ . Then  $D.(K_X + D) \geq -2$  and the equality holds if and only if  $D$  is a rational tree.*

*Proof.* Note that  $\sum_{k < j} D_k.D_j \geq n - 1$  and the equality holds if and only if  $D$  is a tree. We calculate:  $D.(K_X + D) = \sum D_j^2 + \sum K_X.D_j + 2 \sum_{k < j} D_k.D_j \geq \sum_j (2p_a(D_j) - 2) + 2(n - 1) \geq -2$ . The lemma follows.  $\square$

**Lemma 2.3.** *Suppose that  $X$  is a nonsingular projective surface with  $\chi(\mathcal{O}_X) = 1$  and  $D (\neq 0)$  a connected reduced divisor such that  $H^0(X, K_X + D) = 0$ . Then the following statements are true.*

- (1)  $D$  is a connected rational tree.
- (2) Suppose further that  $D$  supports a nef and big divisor (so  $D$  is automatically connected). Then  $\pi_1(X) = (1)$ .

*Proof.* The Serre duality and Riemann-Roch theorem imply  $0 = h^0(X, K_X + D) = h^1(X, K_X + D) + \frac{1}{2}(K_X + D).D + \chi(\mathcal{O}_X) \geq 0 + (-1) + 1$  by Lemma 2.2. Thus  $D.(K_X + D) = -2$  and hence  $D$  is a connected rational tree by the same lemma. So  $\pi_1(D) = (1)$ . Suppose that  $D$  supports a nef and big effective divisor. Then the surjective map  $\pi_1(D) \rightarrow \pi_1(X)$  in Nori [No, Cor. 2.3] infers  $\pi_1(X) = (1)$ .  $\square$

The next result is a very important restriction on  $X$  and  $L$  in Theorem 1.2.

**Proposition 2.4.** *Let  $X$  be a nonsingular projective surface with  $q(X) = 0$  and  $L$  a nef and big effective  $\mathbf{Q}$ -divisor such that  $H^0(X, K_X + L_{\text{red}}) = 0$ . Then  $\chi(\mathcal{O}_X) = 1$ ,  $L_{\text{red}}$  is a connected rational tree and  $X$  is simply connected.*

*Proof.* Note that  $p_g(X) \leq h^0(X, K_X + L_{\text{red}}) = 0$ . So  $\chi(\mathcal{O}_X) = 1$ . Now the proposition follows from Lemma 2.3.  $\square$

The result below is used in the subsequent sections.

**Lemma 2.5.** *Suppose that  $X$  is a minimal nonsingular projective surface with Kodaira dimension  $\kappa(X) = 1$ ,  $p_g(X) = 0$ , and  $\pi_1^{\text{alg}}(X) = (1)$  (this is true if  $\pi_1(X) = (1)$ ). Let  $\pi : X \rightarrow \mathbf{P}^1$  be the unique elliptic fibration with  $F$  a general fibre. The following statements are true:*

- (1)  $\pi$  has exactly two multiple fibres  $F_1, F_2$ , and their multiplicities  $m_1, m_2$  are coprime. In particular, if  $E$  is horizontal then  $E.F = m_1 m_2 m_3 (\geq 6)$  for some positive integer  $m_3$ .
- (2) Suppose further that a reduced connected divisor  $D$  on  $X$  is a rational tree and contains strictly the support of an effective  $\Gamma$  of elliptic fibre type. Then  $\Gamma$  is a full fibre of  $\pi$  and of type  $II^*$ ,  $(m_1, m_2) = (2, 3)$  and  $E.F = 6$  for some  $E$  in  $D$ .

*Proof.* (1) Since  $\pi_1(X)^{\text{alg}} = (1)$ , we have  $H_1(X, \mathbf{Z}) = (0)$  and hence  $q(X) = 0$ . So  $\chi(\mathcal{O}_X) = 1$ . Since  $\kappa(X) = 1$ , there is an elliptic fibration  $\pi : X \rightarrow \pi(X) = \mathbf{P}^1$ , where the image is  $\mathbf{P}^1$  because  $q(X) = 0$ . Let  $F_i$  ( $1 \leq i \leq t$ ) be all multiple fibres of  $\pi$ , with multiplicity  $m_i$ . If  $m = \gcd(m_1, m_2) \geq 2$ , then the relation  $m(F_1/m - F_2/m) \sim 0$  induces an unramified Galois  $\mathbf{Z}/(m)$ -cover of  $X$ , contradicting the assumption  $\pi_1(X)^{\text{alg}} = (1)$ . If  $t \geq 3$ , then by Fox's solution to Fenchel's conjecture (see [Fo], [Ch]), there is a base change  $B \rightarrow \mathbf{P}^1$  ramified exactly over  $\pi(F_i)$  ( $1 \leq i \leq t$ ) and with ramification index  $m_i$ . Then the normalization  $Y$  of the fibre product  $X \times_{\mathbf{P}^1} B$  is an unramified cover of  $X$  (so that the induced fibration  $Y \rightarrow B$  has no multiple fibres), again contradicting the assumption that  $\pi_1(X)^{\text{alg}} = (1)$ .

On the other hand, by the canonical divisor formula, we have  $K_X = \pi^*(K_{\mathbf{P}^1}) + \chi(\mathcal{O}_X)F_1 + \sum_{i=1}^t (m_i - 1)(F_i)_{\text{red}} \sim_{\mathbf{Q}} (-1 + \sum_{i=1}^t (1 - \frac{1}{m_i}))F_1$  (so  $\pi$  is the only elliptic fibration on  $X$ ). Since  $\kappa(X) = 1$ , we see that  $t \geq 2$ . Now the lemma follows from the results above.

(2) Since  $\Gamma$  is of elliptic fibre type,  $0 = K_X \cdot \Gamma = \Gamma^2 = 0$ . Hence  $\Gamma$  is a multiple of a fibre of  $\pi$ . Since the support of  $\Gamma$  ( $< D$ ) is a tree, it is of type  $I_n^*$ ,  $II^*$ ,  $III^*$  or  $IV^*$ , whence  $\Gamma$  is a full fibre (and is not a multiple fibre). By the assumption, there is an  $E$  in  $D$  such that  $\text{Supp}(E + \Gamma)$  is a connected rational tree. Thus  $E \cdot \Gamma \leq 6$  and the equality holds if and only if  $\Gamma$  is of type  $II^*$  and  $E$  meets the coefficient-6 component of  $\Gamma$ . Now (2) follows from (1).  $\square$

The example below shows that an assumption like the condition (3) in Conjecture 1.1 might be necessary.

**Example 2.6.** We shall construct a nonsingular projective surface  $X$  and a  $\mathbf{Q}$ -divisor  $L$  such that the conditions (1) and (2) in Conjecture 1.1 are satisfied, but that  $H^0(X, K_X + \lceil L \rceil) = 0$ . Indeed, we will see that both  $L$  and  $K_X + L$  are nef and big  $\mathbf{Q}$ -divisors.

Let  $C$  be a sextic plane curve with 9 ordinary cusps (of type  $(2, 3)$ ) and no other singularities. This  $C$  (regarded as a curve in the dual plane  $\mathbf{P}^{2*}$ ) is dual to a smooth plane cubic (always having 9 inflectins). Let  $\bar{X} \rightarrow \mathbf{P}^2$  be the double cover branched at  $C$ . Then  $\bar{X}$  is a normal  $K3$  surface with exactly 9 Du Val singularities (lying over the 9 cusps) of Dynkin type  $A_2$ . Let  $X$  be the minimal resolution. According to Barth [Ba], these  $9A_2$  are 3-divisible. That is, for some integral divisor  $G$ , we have  $3G \sim \sum_{i=1}^9 (C_i + 2D_i)$  where  $\coprod (C_i + D_i)$  is a disjoint union of the 9 intersecting  $\mathbf{P}^1$  (i.e., the  $9A_2$ ). Let  $H$  be the pull back of a general line away from the 9 cusps on  $C$ . Then  $H^2 = 2$  and  $H$  is disjoint from the  $9A_2$ , so  $H \cdot G = 0$ . We can also calculate that  $G^2 = -6$ . Now let  $L = H + G - \frac{1}{3} \sum_{i=1}^9 (C_i + 2D_i)$ . Then  $\lceil L \rceil = H + G$  and  $\lceil L \rceil^2 = -4$ . Clearly,  $K_X + L = L \equiv H$  is nef and big. However, by the Kawamata-Viehweg vanishing, and Riemann-Roch theorem, we have  $h^0(X, K_X + \lceil L \rceil) = \frac{1}{2} \lceil L \rceil^2 + 2 = 0$ .

A similar example can be constructed, if one can find a quartic surface with 16 nodes (i.e., a normal Kummer quartic surface).

### 3. IRREGULAR SURFACES

In this section, we shall show that Conjecture 1.1 holds true (with only the first two conditions there but not the last condition) for surfaces  $X$  with positive irregularity  $q(X)$ .

To be precise, let  $X$  be a nonsingular projective surface with  $q(X) > 0$  and let  $\text{alb} : X \rightarrow \text{Alb}(X)$  be the Albanese map. Then we have:

**Theorem 3.1.** *Let  $X$  be a nonsingular projective surface with  $q(X) > 0$ . Let  $L$  be a nef and big  $\mathbf{Q}$ -divisor such that  $K_X + L$  is nef. Then  $H^0(X, K_X + \lceil L \rceil) \neq 0$ .*

To see this, we need the following lemma:

**Lemma 3.2.** *Let  $\mathcal{F} \neq 0$  be a  $IT^0$  sheaf on an abelian variety  $A$ , i.e. for every  $i > 0$  we have  $H^i(A, \mathcal{F} \otimes P) = 0$  for all  $P \in \text{Pic}^0(A)$ . Then  $H^0(A, \mathcal{F}) \neq 0$ .*

The proof can be found in [CH], but we reprove it here.

*Proof.* Suppose on the contrary that  $H^0(A, \mathcal{F}) = 0$ . Since  $\mathcal{F}$  is  $IT^0$ , the Fourier-Mukai transform of  $\mathcal{F}$  is a locally free sheaf of rank  $= h^0(A, \mathcal{F})$ , hence the zero sheaf. The only sheaf that transforms to the zero sheaf is the zero sheaf, which is a contradiction.  $\square$

*Proof of Theorem 3.1.* Let  $f : X' \rightarrow X$  be an embedded resolution for  $(X, L)$ . It is clear that  $f^*L$  is nef and big with simple normal crossing support. Let  $\Delta := \lceil f^*L \rceil - f^*L$ , then  $\Delta$  is klt. By a property of nef and big divisor (see e.g. [La], ex 2.2.17), there is an effective divisor  $N$  such that  $A_k := f^*L - \frac{1}{k}N$  is ample for all  $k \gg 0$ . We fix  $k$  such that  $\Delta + \frac{1}{k}N$  is klt. Now we can write  $A_k = (\text{alb} \circ f)^*M + E$  for some ample  $\mathbf{Q}$ -divisor  $M$  on  $A := \text{Alb}(X)$  and effective divisor  $E$  on  $X'$ . Pick irreducible divisor  $B \in |(n-1)A|$  for  $n \gg 0$  such that  $\Delta' := \Delta + \frac{1}{k}N + \frac{1}{n}E + \frac{1}{n}B$  is again klt. Then we have, where  $P' = (\text{alb} \circ f)^*P$  with  $P \in \text{Pic}^0(A)$ :

$$K_{X'} + \lceil f^*L \rceil \otimes P' \equiv K_{X'} + \frac{(\text{alb} \circ f)^*M}{n} + \Delta'.$$

Let  $\mathcal{F} := \text{alb}_* f_* \mathcal{O}_{X'}(K_{X'} + \lceil f^*L \rceil)$ . By Kollár's relative vanishing theorem (cf. [Ko], 10.19.2), one sees that  $\mathcal{F}$  is  $IT^0$ .

We claim that  $\mathcal{F} \neq 0$ .

Grant this claim for the time being. By the above lemma, it follows that

$$h^0(X', K_{X'} + \lceil f^*L \rceil) = h^0(A, \mathcal{F}) \neq 0.$$

Since  $K_{X'} + \lceil f^*L \rceil = f^*(K_X + \lceil L \rceil) + \Delta$ , where  $\Delta$  is an exceptional divisor (possibly non-effective). It's easy to see that  $f_* \mathcal{O}_{X'}(\Delta) \subset \mathcal{O}_X$ . By the projection formula, one has:

$$0 \neq H^0(X', K_{X'} + \lceil f^*L \rceil) = H^0(X, K_X + \lceil L \rceil) \otimes f_* \mathcal{O}_{X'}(\Delta) \subset H^0(X, K_X + \lceil L \rceil).$$

This is the required non-vanishing.

To see the claim, if  $\dim(\text{alb}(X)) = 2$ , then  $\text{alb} \circ f$  is generically finite. Hence it is clear that  $\mathcal{F} \neq 0$ . If  $\dim(\text{alb}(X)) = 1$ . Let  $F$  be a general fiber of  $\text{alb} \circ f$ . Then we have:

$$\text{rank}(\mathcal{F}) = h^0(F, (K_{X'} + \lceil f^*L \rceil)|_F) = h^0(F, K_F + \lceil f^*L|_F \rceil).$$

Since  $f^*L$  is big,  $f^*L.F > 0$ . It follows that  $\text{deg}(\lceil f^*L|_F \rceil) > 0$ .

If  $g(F) > 0$ , then we have  $h^0(F, K_F + \lceil f^*L|_F \rceil) > 0$  already. If  $g(F) = 0$ , note that  $K_X + L$  is nef. Note also that  $(K_{X'} + f^*L).F = (K_X + L).f(F)$  since  $F$  is general. This implies that

$$\begin{aligned} \text{deg}(K_F + \lceil f^*L|_F \rceil) &= (K_{X'} + f^*L).F + (\lceil f^*L \rceil - f^*L).F \\ &= (K_X + L).f(F) + (\lceil L \rceil - L).F \geq 0. \end{aligned}$$

Hence  $h^0(F, K_F + \lceil L_F \rceil) > 0$ . We conclude that  $\mathcal{F} \neq 0$  and hence the required non-vanishing that  $h^0(X, K_X + \lceil L \rceil) \neq 0$ .  $\square$

*Remark 3.3.* In the proof of Theorem 3.1, without taking log-resolution at the beginning, one can apply Sakai's lemma [Sa] for surfaces to get the vanishing of higher cohomology. However, our argument here works for higher dimensional situation. It shows that non-vanishing for general fiber gives the non-vanishing.

#### 4. SURFACES OF KODAIRA DIMENSION 0

In this section, we show that the Conjecture 1.1 in the Introduction is true for surfaces  $X$  (not necessarily minimal) with Kodaira dimension  $\kappa(X) = 0$ .

**Theorem 4.1.** *Suppose that  $X$  is a nonsingular projective surface (not necessarily minimal) of Kodaira dimension  $\kappa(X) = 0$ . Then Conjecture 1.1 is true (for effective  $\mathbf{Q}$ -divisor  $L$ ).*

*Proof.* By Theorem 3.1, we may assume that  $q(X) = 0$ . We may also assume that  $0 = h^0(X, K_X + L_{\text{red}}) (\geq p_g(X))$ . So  $X$  is the blow up of an Enriques surface by the classification theory. On the other hand,  $\pi_1(X) = (1)$  by Proposition 2.4, a contradiction. This proves the theorem.  $\square$

#### 5. SURFACES WITH NEGATIVE $\kappa$ , PART I : RULED SURFACES

In this section, we prove Conjecture 1.1 for relatively minimal surfaces  $X$  of Kodaira dimension  $\kappa(X) = -\infty$ . By Theorem 3.1, we may assume that  $q(X) = 0$ , so  $X$  is a relatively minimal rational surface. If  $X = \mathbf{P}^2$  or  $\mathbf{P}^1 \times \mathbf{P}^1$ , it is easy to verify that Conjecture 1.1 is true since effective divisor is then nef. We thus assume that  $X$  is the Hirzebruch surface  $\mathbb{F}_d$  of degree  $d \geq 1$  (though,  $\mathbb{F}_1$  is not relatively minimal).

We first fix some notations. Let  $\pi : \mathbb{F}_d \rightarrow \mathbf{P}^1$  be the ruling. Let  $F$  be a general fibre and  $C$  the only negative curve (a cross-section, indeed) on  $\mathbb{F}_d$ . So  $C^2 = -d$ .

**Theorem 5.1.** *Let  $X$  be a relative minimal surface of Kodaira dimension  $\kappa(X) = -\infty$ . Then Conjecture 1.1 holds (for effective  $\mathbf{Q}$ -divisor  $L$ ).*

*Proof.* As mentioned above, we assume that  $X = \mathbb{F}_d$  for some  $d \geq 1$ . Let  $L$  be a nef and big effective  $\mathbf{Q}$ -divisor such that  $K_X + L$  is nef. If  $\text{Supp}(L)$  does not contain the negative curve  $C$ , then  $E := \lceil L \rceil - L$  is effective and nef; so  $\lceil L \rceil = L + E$  is nef and big and  $K_X + \lceil L \rceil = K_X + L + E$  is nef; then the Serre duality and Riemann-Roch theorem for Cartier divisor imply that  $h^0(X, K_X + \lceil L \rceil) \geq \frac{1}{2} \lceil L \rceil (K_X + \lceil L \rceil) + \chi(\mathcal{O}_X) \geq 0 + 1$ . Therefore, we may assume that  $\text{Supp}(L)$  contains  $C$ .

Write  $L = \sum_i c_i C_i + \sum f_j F_j$  where  $C_1 = C$ , the  $C_i$ 's are distinct horizontal components and  $F_j$ 's are distinct fibres, where  $c_i > 0$ ,  $f_j > 0$ .

Suppose on the contrary that  $H^0(X, K_X + \lceil L \rceil) = 0$ . Then by Lemma 2.3,  $L_{\text{red}}$  is a connected rational tree. Hence one of the following cases occurs:

Case (i).  $L = c_1 C_1 + \sum_{j=1}^k f_j F_j$  ( $k \geq 0$ ),

Case (ii).  $L = \sum_{i=1}^k c_i C_i + f_1 F_1$  ( $k \geq 2$ ), and  $L_{\text{red}}$  is comb-shaped, i.e.,  $C_i$ 's are disjoint cross-sections.

Case (iii).  $L = \sum_{i=1}^k c_i C_i$  ( $k \geq 2$ ).

Recall that  $K_X \sim -2C_1 - (d+2)F$ . The nefness of  $K_X + L$  implies:

$$\begin{aligned} 0 &\leq (K_X + L).F = -2 + \sum c_i(C_i.F), \\ 0 &\leq (K_X + L).C_1 = d - 2 - dc_1 + \sum_{i \geq 2} c_i(C_i.C_1) + \sum f_j, \\ &\sum c_i(C_i.F) \geq 2, \\ &\sum f_j \geq 2 + (c_1 - 1)d - \sum_{i \geq 2} c_i(C_i.C_1). \end{aligned}$$

In Case (i), the above inequalities imply  $c_1 \geq 2$  and  $\sum f_j \geq 2 + (c_1 - 1)d \geq d + 2$ , whence  $[L] = [c_1]C_1 + \sum [f_j]F_j \sim [c_1]C_1 + (\sum [f_j])F \geq 2C_1 + (\sum f_j)F \geq 2C_1 + (d+2)F_1 \sim -K_X$ . Hence  $H^0(X, K_X + [L]) \neq 0$ .

Consider Case (ii). Then one sees easily that  $k = 2$  and  $C_2 \sim C_1 + dF_1$  (see [Ha, Chapter V, §2]). By the displayed inequalities, we have  $c_1 \geq 2 - c_2$  and  $f_1 \geq 2 + (c_1 - 1)d$ . If  $c_2 > 1$  then  $[L] \geq C_1 + 2C_2 + F_1 > -K_X$ , whence  $H^0(X, K_X + [L]) \neq 0$ . So we may assume that  $c_2 \leq 1$ . Then  $c_1 \geq 1$  and  $f_1 \geq 2$ . Thus  $[L] \geq C_1 + C_2 + 2F_1 \sim -K_X$ , whence  $H^0(X, K_X + [L]) \neq 0$ .

Consider Case (iii). Since  $L$  is a connected tree, we may assume that  $C_1.C_2 = 1$ . So  $C_2 \sim n(C_1 + dF) + F$  for some integer  $n \geq 1$ . Since  $C_i$  ( $i \geq 2$ ) is irreducible, we have  $C_i \geq C_1 + dF$  by [Ha]. If  $k \geq 3$  or  $n \geq 2$ , then we see that  $[L] \geq \sum_{i=1}^k C_i > -K_X$ . So assume that  $k = 2$  and  $n = 1$ . By the inequalities displayed above, we have  $c_1 \geq 2 - c_2$  and  $c_2 \geq 2 + (c_1 - 1)d$ . If  $c_2 > 1$  then  $[L] \geq C_1 + 2C_2 > -K_X$ . So assume that  $c_2 \leq 1$ . Then  $c_1 \geq 2 - 1$  and  $c_2 \geq 2 + 0d$ , a contradiction.  $\square$

## 6. SURFACES WITH NEGATIVE $\kappa$ , PART II: RELATIVELY MINIMAL ELLIPTIC

In this section we consider relatively minimal elliptic surface  $\pi : X \rightarrow B$  with Kodaira dimension  $\kappa(X) = -\infty$ . As far as the Conjecture 1.1 is concerned, we may assume that the irregularity  $q(X) = 0$  by virtue of Theorem 3.1. So  $X$  is a rational surface and  $B = \mathbf{P}^1$ . By the canonical divisor formula, we see that  $\pi$  has at most one fibre  $F_0$  with multiplicity  $m \geq 2$ ; moreover, such  $F_0$  (if exists) is of Kodaira type  $I_n$  ( $n \geq 0$ ), and  $-K_X = (F_0)_{\text{red}}$ .

We show that Conjecture 1.1 is true if  $K_X + L$  is nef and big (but without the assumption of the effectiveness of  $L$ ):

**Theorem 6.1.** *Let  $\pi : X \rightarrow B$  be a relatively minimal elliptic surface with  $\kappa(X) = -\infty$ . Suppose that  $L$  is a nef and big  $\mathbf{Q}$ -divisor such that  $K_X + L$  is nef and big. Then  $H^0(X, K_X + [L]) \neq 0$ .*

*Proof.* By Theorem 3.1, we may assume that  $q(X) = 0$ , so  $B = \mathbf{P}^1$  and  $X$  is a rational surface.

Suppose that the  $\mathbf{Q}$ -divisor  $L$  is nef and big and  $K_X + L$  is nef. Let  $F_0 = m(F_0)_{\text{red}}$  be the multiple fiber. We set  $m = 1$  and let  $F_0$  be a general (smooth) fibre, if  $\pi$  is multiple fibre free. Then  $K_X \sim -(F_0)_{\text{red}}$ . Let  $a > 0$ . Consider the exact sequence:  $0 \rightarrow \mathcal{O}_X(K_X + [aL] - (F_0)_{\text{red}}) \rightarrow \mathcal{O}_X(K_X + [aL]) \rightarrow \mathcal{O}_{(F_0)_{\text{red}}}(K_X + [aL]|_{(F_0)_{\text{red}}}) \rightarrow 0$ .

Let us find the condition for  $aL - (F_0)_{\text{red}}$  to be nef and big. Note that  $aL - (F_0)_{\text{red}} \sim aL + K_X = (a-1)L + (K_X + L)$ . So  $aL - (F_0)_{\text{red}}$  is nef and big if either

- (i)  $a > 1$ , or

(ii)  $a = 1$  and  $K_X + L$  is nef and big.

Assume that either (i) or (ii) is satisfied. Then  $H^1(X, K_X + [aL] - (F_0)_{\text{red}}) = 0 = H^1(X, K_X + [aL])$  by Sakai's vanishing for surfaces. For the integral divisor  $M := K_X + [aL]$  and the reduced divisor  $C := (F_0)_{\text{red}}$  on  $X$ , the above exact sequence implies that  $\chi(\mathcal{O}_C(M|C)) = \chi(\mathcal{O}_X(M)) - \chi(\mathcal{O}_X(M - C)) = C.M - C.(K_X + C)/2$ , where we applied the Riemann-Roch theorem for both  $\mathcal{O}_X(M)$  and  $\mathcal{O}_X(M - C)$ . Now  $C.(K_X + C) = 0$  and  $C.M \geq (F_0)_{\text{red}}.(K_X + aL) > 0$  (for  $0 \neq C$  being nef and  $K_X + aL$  nef and big), so  $\chi(\mathcal{O}_C(M|C)) > 0$ . By the vanishing above,  $h^0(X, K_X + [aL]) = \chi(\mathcal{O}_X(M)) = \chi(\mathcal{O}_X(M - C)) + \chi(\mathcal{O}_C(M|C)) = h^0(X, K_X + [aL] - (F_0)_{\text{red}}) + \chi(\mathcal{O}_C(M|C)) > 0 + 0$ . This proves the theorem.  $\square$

*Remark 6.2.* The above argument actually proved the following: let  $\pi : X \rightarrow B$  be a relatively minimal elliptic surface with  $\kappa(X) = -\infty$ . Suppose that  $L$  is a nef and big  $\mathbf{Q}$ -divisor such that  $K_X + L$  is nef. Then  $H^0(X, K_X + [aL]) \neq 0$  provided that either  $a > 1$ , or  $a = 1$  and  $K_X + L$  is nef and big.

## 7. PREPARATIONS FOR SURFACES WITH $\kappa = 1$ OR 2

Throughout this section, we assume that  $X$  is a nonsingular projective surface with  $K_X$  nef and Kodaira dimension  $\kappa(X) = 1$  or 2. The main result is Proposition 7.10 to be used in the next section.

**Definition 7.1.** (1) Let  $\Gamma$  be a connected rational tree on  $X$ . This  $\Gamma$  is of type  $A'_n$  (resp.  $D'_n$ , or  $E'_n$ ) if its weighted dual graph is of Dynkin type  $A_n$  (resp.  $D_n$ , or  $E_n$ ) but its weights may not all be  $(-2)$ .

(2) Let  $\Gamma$  be a connected effective integral divisor on  $X$  so that  $\text{Supp}(\Gamma)$  is a connected rational tree.  $\Gamma$  is of type  $I_n^*$  (resp.  $II^*$ , or  $III^*$ , or  $IV^*$ ) if  $\Gamma$  is of the respective elliptic fibre type (hence  $\text{Supp}(\Gamma)$  is a union of  $(-2)$ -curves).  $\Gamma$  is of type  $I_n^{*'} (resp. II^{*'}$ , or  $III^{*'}$ , or  $IV^{*'}$ ) if  $\Gamma$  is equal to an elliptic fibre of type  $I_n^*$  (resp.  $II^*$ , or  $III^*$ , or  $IV^*$ ), including coefficients, but the self intersections of components of  $\Gamma$  may not all be  $(-2)$ . E.g.  $\Gamma = 2 \sum_{i=0}^n C_i + \sum_{j=n+1}^{n+4} C_j$  is of type  $I_n^{*'}$ , where  $C_i + C_0 + C_1 + \dots + C_n + C_j$  is an ordered linear chain for all  $i \in \{n+1, n+2\}$  and  $j \in \{n+3, n+4\}$ .

The assertion(1) below follows from  $C^2 = -2 - C.K_X \leq -2$ . The others are clear.

**Lemma 7.2.** (1) *If  $C$  is a smooth rational curve on  $X$ , then  $C^2 \leq -2$ .*

(2) *If  $\Gamma$  is of type  $A_n'$ ,  $D_n'$  or  $E_n'$  then it is negative definite, i.e., the intersection matrix of components in  $\Gamma$  is negative definite.*

(3) *If  $\Gamma$  is one of types  $I_n^*$ ,  $II^*$ ,  $III^*$  and  $IV^*$  (resp.  $I_n^{*'}$ ,  $II^{*'}$ ,  $III^{*'}$  and  $IV^{*'}$ ), but at least one component of  $\Gamma$  is not a  $(-2)$ -curve, then  $\Gamma$  is negative semi-definite (resp. negative definite).*

(4) *If  $\Gamma$  is a connected rational tree with (the number of irreducible components)  $\#\Gamma \leq 5$ , then  $\Gamma$  is negative definite, unless  $\Gamma$  supports a divisor of type  $I_0^*$ .*

The Picard number can be estimated in the following way:

**Lemma 7.3.** *Let  $\Gamma$  be a connected rational tree whose  $(-2)$ -components do not support a divisor of type  $I_0^*$ . Let  $r = \min\{9, \#\Gamma - 1\}$ . Then there is a subgraph  $\Gamma'$  of  $r$  components with negative definite intersection matrix. In particular,  $\rho(X) \geq r + 1$ . Also if  $\rho(X) \leq 9$  then  $\#\Gamma \leq 9$ .*

*Proof.* We have only to prove the first assertion. By taking a subgraph, we may assume that  $\#\Gamma \leq 10$ .

If  $\Gamma$  is a linear chain, then it has negative definite intersection matrix, and we are done. Thus we may assume that there exists an irreducible component which meets more than two other irreducible components. Let  $C_0$  be the irreducible component that meets  $k$  other components with the largest  $k$ . Then  $\Gamma - C_0$  has exactly  $k$  connected components  $\{\Delta_i\}$ . We may assume that  $k \geq 3$ . Let  $C_i$  be the irreducible component of  $\Delta_i$  that meets  $C_0$ .

By Lemma 7.2, if  $\#\Delta_i \leq 5$  for all  $i$  then each  $\Delta_i$  is negative definite. By taking  $\Gamma' = \sum \Delta_i$ , we are done.

The remaining cases of  $(\#\Delta_1, \dots, \#\Delta_k)$  are  $\{(1, 1, 6), (1, 1, 7), (1, 2, 6), (1, 1, 1, 6)\}$ . For the case  $(1, 1, 1, 6)$ , we take  $\Gamma' = \Gamma - C_4$ , then now  $\Gamma'$  has at least two connected components:  $C_0 + C_1 + C_2 + C_3$  and others. It is clear that each connected component has at most 5 irreducible components. Hence  $\Gamma'$  is negative definite. For the cases  $(1, 1, 6)$  and  $(1, 2, 6)$ , similar argument works.

It remains to work with the case  $(1, 1, 7)$ . If  $C_3$  meets at least 3 components, we take  $\Gamma' = \Gamma - C_3$ . Then  $\Gamma'$  has at least 3 connected components and each one has length  $\leq 5$ . If  $C_3$  meets 2 components, say  $C_0, C_4$ , then we take  $\Gamma' = \Gamma - C_4$ . Again, each connected component of  $\Gamma'$  has at most 5 irreducible components. This proves the lemma.  $\square$

**Lemma 7.4.** *Suppose that  $q(X) = p_g(X) = 0$  and  $\pi_1^{\text{alg}}(X) = (1)$  (these are satisfied in the situation of Proposition 7.10; see its proof).*

- (1) *We have  $\rho(X) \leq 10 - K_X^2 \leq 10$ , and  $\rho(X) = 10$  holds only when  $\kappa(X) = 1$ .*
- (2) *For  $L$  in Proposition 7.10, suppose that some  $(-2)$ -components of  $L$  support an effective divisor  $\Gamma$  of elliptic fibre type. Then  $\Gamma$  is of type  $II^*$ ,  $\kappa(X) = 1$  and  $\rho(X) = 10 \leq \#L$ . Moreover,  $L_{\text{red}}$  supports an effective divisor  $C$  of type  $I_0^*$  whose central and three of the tip components are all  $(-2)$ -curves.*

*Proof.* (1) follows from:  $\rho(X) \leq b_2(X) = c_2(X) - 2 + 4q(X) = 12\chi(\mathcal{O}_X) - K_X^2 - 2 = 10 - K_X^2 \leq 10$  (Noether's equality).

(2) Since a surface of general type does not contain such  $\Gamma$ , we have  $\kappa(X) = 1$ . By Lemma 2.5 and its notation and noting that  $L_{\text{red}} > \text{Supp}(\Gamma)$  (for  $L$  being nef and big),  $\Gamma$  is of type  $II^*$  and  $\text{Supp}(E + \Gamma) (\leq L_{\text{red}})$  supports a  $I_0^*$  as described in (2). Also  $\#L \geq \#\Gamma + 1 = 10$  and  $\rho(X) \geq 2 + (\#\Gamma - 1) = 10$ . Thus  $\rho(X) = 10$ . This proves the lemma.  $\square$

By the lemma above and Lemma 7.3, to prove Proposition 7.10, we may assume:

*Remark 7.5.* Assumption:  $\#L \leq 9$ , and the  $(-2)$ -components of  $L$  do not support a divisor of elliptic fibre type.

We need three more lemmas in proving Proposition 7.10.

**Lemma 7.6.** *Let  $D = \sum_{i=0}^n D_i$  be a reduced divisor on  $X$ . Suppose that  $D - D_0$  has a negative definite  $n \times n$  intersection matrix  $(D_i \cdot D_j)_{1 \leq i, j \leq n}$  and  $D$  supports a divisor with positive self intersection.*

- (1) *We have  $\det(D_k \cdot D_\ell)_{0 \leq k, \ell \leq n} > 0$  (resp.  $< 0$ ) if  $n$  is even (resp. odd).*
- (2) *Assign formally  $G_i := D_i$  and define  $G_i \cdot G_j := D_i \cdot D_j$  ( $i \neq j$ ) and  $G_i^2 := -x_i$ . Suppose that (\*) the  $n \times n$  matrix  $(G_i \cdot G_j)_{1 \leq i, j \leq n}$  is negative definite. If  $G_i^2 \geq D_i^2$  for all  $0 \leq i \leq n$ , then (\*\*)  $\det(G_k \cdot G_\ell)_{0 \leq k, \ell \leq n} > 0$  (resp.  $< 0$ ) if  $n$  is even (resp. odd).*

(3) Suppose that  $D_i^2 \leq -2$  for all  $0 \leq i \leq n$ . In (2) above for  $0 \leq k \leq n$ , choose the largest positive integer  $m_k$  (if exists) such that (\*) and (\*\*) in (2) are satisfied for  $G_i$  with  $G_k^2 = -m_k$  and  $G_i^2 = -2$  ( $i \neq k$ ). Then  $D_k^2 \geq -m_k$ .

*Proof.* For (1), suppose that the matrix in (1) is similar (over  $\mathbf{Q}$ ) to a diagonal matrix  $J$ . Then the condition implies that  $J$  has one positive and  $n$  negative diagonal entries. So (1) follows.

For (2), we have only to show that a linear combination of  $G_i$  has positive self intersection. By the assumption some divisor  $\Delta = \sum b_i D_i$  has positive self intersection, then so is  $\Gamma = \sum b_i G_i$  because  $\Gamma^2 = \sum b_i b_j G_i \cdot G_j \geq \sum b_i b_j D_i \cdot D_j = \Delta^2 > 0$ . The (3) follows from (2).  $\square$

Let  $D$  be a reduced divisor and let  $D = P + N$  be the Zariski decomposition with  $P$  the nef and  $N$  the negative part so that  $P$  and  $N$  are effective  $\mathbf{Q}$ -divisor with  $P \cdot N = 0$  (see [Fu1], [Fu2], [Mi]).  $D$  supports a nef and big divisor if and only if  $P^2 > 0$ .

In Lemmas 7.7 and 7.8 below, we do not need the bigness of  $P$ ; in Lemma 7.7,  $K_X$  is irrelevant.

**Lemma 7.7.** (1) Write  $P = \sum_{i=0}^n p_i D_i$ . Then  $0 \leq p_i \leq 1$ , and  $p_i < 1$  holds if and only if  $D_i \leq \text{Supp}(N)$ .

(2) Write  $\text{Supp}(N) = \sum_{i=0}^s D_i$  after relabelling. Then  $(p_0, \dots, p_s)$  is the unique solution of the linear system  $\sum_{i=0}^n x_i (D_i \cdot D_j) = 0$  ( $j = 0, \dots, s$ ), where we set  $x_j = 1$  ( $j > s$ ).

(3) Assign formally  $G_i := D_i$  and  $G_i \cdot G_j = D_i \cdot D_j$  ( $i \neq j$ ). Suppose that for  $\alpha \leq i \leq \beta$ , we assign  $G_i^2$  such that  $-2 \geq G_i^2 \geq C_i^2$  and  $(G_i \cdot G_j)_{\alpha \leq i, j \leq \beta}$  is negative definite. Let  $(x_i = b_i | \alpha \leq i \leq \beta)$  be the unique solution of the linear system  $\sum_{i=0}^n x_i G_i \cdot G_j = 0$  ( $\alpha \leq j \leq \beta$ ), where we set  $x_j = b_j = p_j$  if  $j < \alpha$  or  $j > \beta$ . Then  $b_i \geq p_i$  for all  $\alpha \leq i \leq \beta$ .

*Proof.* For (1), see [Fu1] or [Mi]. (2) follows from the fact that  $P \cdot D_j = 0$  ( $1 \leq j \leq s$ ) and that  $N$  has negative definite (and hence invertible) intersection matrix.

We prove (3). It suffices to show that (\*) the sum  $\sum_{i=\alpha}^{\beta} (b_i - p_i) G_i \cdot G_j \leq 0$  for all  $\alpha \leq j \leq \beta$ .

Indeed, write  $\sum (b_i - p_i) G_i = A - B$  with  $A \geq 0, B \geq 0$  and with no common components in  $A$  and  $B$ ; then the condition (\*) implies that  $A \cdot B - B^2 = \sum (b_i - p_i) G_i \cdot B \leq 0$ ; this and  $A \cdot B \geq 0$  and  $B^2 \leq 0$  imply that  $B^2 = 0$  and hence  $B = 0$  by the negative-definiteness of  $(G_i \cdot G_j)$ .

Coming to the sum in (\*) above, it is equal to  $\sum_{i=0}^n b_i G_i \cdot G_j - \sum_{i=0}^n p_i G_i \cdot G_j \leq 0 - \sum_{i=0}^n p_i D_i \cdot D_j = 0$ . This proves the lemma.  $\square$

**Lemma 7.8.** Suppose that  $\Gamma = D_1 + \dots + D_m$  is an ordered linear chain contained in  $D$  such that  $\Gamma \cdot (D - \Gamma) = 1$ . Let  $D_t \leq \Gamma$  and  $D_{m+1} \leq D - \Gamma$  such that  $D_t \cdot D_{m+1} = 1$ . If either  $t = m$  or  $D_t^2 \leq -3$ , then  $\Gamma \leq \text{Supp}(N)$ .

*Proof.* Write  $P = \sum_j p_j D_j$ . If  $t = m$ , we set  $G_i^2 = -2$  ( $1 \leq i \leq m$ ) in Lemma 7.7 and obtain  $p_i \leq b_i = (i/(m+1))p_{m+1} < 1$  and hence  $\Gamma \leq \text{Supp}(N)$ . If  $D_t^2 \leq -3$ , we have only to show that  $p_t < 1$  because we already have  $p_j < 1$  for every  $1 \leq j \leq m$  with  $j \neq t$ , by the previous case. Now  $0 \leq P \cdot D_t = p_t D_t^2 + p_{t-1} + p_{t+1} + p_{m+1} < -3p_t + 3$ , whence  $p_t < 1$ . This proves the lemma.  $\square$

For  $L$  in Proposition 7.10, let  $L_{\text{red}} = P + N$  be the Zariski decomposition, so  $P \geq 0$  and  $N \geq 0$ . By the maximality of  $P$ , we have  $L_{\text{red}} \geq P \geq L$ . So  $P_{\text{red}} = L_{\text{red}}$ . Write

$$P = \sum_{i=0}^n p_i C_i,$$

Then  $0 < p_i \leq 1$ . Note that  $p_j = 1$  for some  $j$  for otherwise  $\text{Supp}(L) = \text{Supp}(P) \subseteq \text{Supp}(N)$  would be negative definite. So we assume the following (after relabelling):

*Remark 7.9.* Additional assumption :  $L = P$  and  $p_0 = 1$  (and also  $\#P \leq 9$ ).

Now we state the main result of the section.

**Proposition 7.10.** *Let  $X$  be a minimal nonsingular projective surface (i.e.,  $K_X$  is nef) with  $p_g(X) = 0$ . Suppose that a  $\mathbf{Q}$ -divisor  $L$  is nef and big and a rational tree. Then  $X$  is simply connected and  $\text{Supp}(L)$  is connected. Moreover, either (the number of irreducible components)  $\#L \geq 10 = \rho(X)$  and  $\kappa(X) = 1$ , or  $\#L \leq 9$  and (A) or (B) below is true:*

(A) *There is a linear chain  $C = \sum_{i=0}^r C_i \leq L_{\text{red}}$  with  $r \geq 0$  (after relabelling) such that  $L_{\text{red}} \cdot \sum_{i=0}^r C_i \geq 2$ .*

(B)  *$\text{Supp}(L)$  supports an effective divisor  $C$  of type in  $\{I_n^*, III^*, IV^*\}$  but the weights of the multiplicity  $\geq 2$  components of  $C$  are all  $(-2)$ , so  $C \cdot (K_X + C) = -2$ . Also the type  $III^*$  occurs only when  $L_{\text{red}}$  is given as follows:*

*Case (B1).  $\kappa(X) = 1$  and  $\rho(X) = 10$ ;  $\det(\text{Pic}(X)) = -1$ , and  $\text{Pic}(X)$  is generated by the divisor class of  $K_X$  and those of the 9 curves in  $L_{\text{red}} = \sum_{i=0}^8 C_i$ ;  $C_0$  meets exactly  $C_1, C_2, C_3$ ;  $C_2 + C_4 + C_6$  and  $C_3 + C_5 + C_7 + C_8$  are linear chains;  $C_6^2 = -3$  and  $C_i^2 = -2$  ( $i \neq 6$ ).*

*Proof.* Since  $L$  is nef and big and a rational tree,  $\kappa(X) = 1, 2$ . Since  $L$  is nef and big, a positive multiple of  $L$  is Cartier and 1-connected. By [No, Cor. 2.3] or the proof of Lemma 2.3,  $\pi_1(X) = (1)$ . In particular,  $q(X) = 0$  and  $\chi(\mathcal{O}_X) = 1$ .

Since  $p_0 = 1$  by the additional assumption,  $C_0$  is not in  $\text{Supp}(N)$ . Since  $0 \leq P \cdot C_0 = C_0^2 + \sum p_j$  and  $C_0^2 \leq -2$ , where  $j$  runs in the set so that  $C_j$  meets  $C_0$ , this  $C_0$  meets at least two components of  $\text{Supp}(P) - C_0$ . Now the proposition follows from the lemmas below.  $\square$

**Lemma 7.11.** *Suppose that  $C_0$  meets exactly two components of  $\text{Supp}(P) - C_0$ . Then Proposition 7.10 is true.*

*Proof.* Suppose that  $C_0$  meets only  $C_1$  and  $C_2$  in  $\text{Supp}(P) - C_0$ . Then  $0 \leq P \cdot C_0 = C_0^2 + p_1 + p_2$  and  $C_0^2 \leq -2$  imply that  $p_1 = p_2 = 1$  and  $C_0^2 = -2$ . Inductively, we can prove that there is an ordered linear chain (after relabelling)  $\sum_{i=a}^b C_i$  in  $\text{Supp}(P)$  such that  $p_i = 1$  and  $C_i^2 = -2$  for all  $a \leq i \leq b$  and  $C_a$  (resp.  $C_b$ ) meets  $C_{a-1}$  and  $C_{a-2}$  (resp.  $C_{b+1}$  and  $C_{b+2}$ ) such that  $C_{a-2} + C_{a-1} + 2 \sum_{i=a}^b C_i + C_{b+1} + C_{b+2}$  is of type  $I_{b-a}^*$  and Proposition 7.10 (B) is true.  $\square$

**Lemma 7.12.** *Suppose that  $C_0$  meets at least four components of  $\text{Supp}(P) - C_0$ . Then Proposition 7.10 is true.*

*Proof.* Suppose that  $C_0$  meets  $C_i$  ( $1 \leq i \leq k$ ) with  $k \geq 4$ . Let  $\Delta_i$  ( $1 \leq i \leq k$ ) be the connected component of  $P_{\text{red}} - C_0$  containing  $C_i$ . Set  $n_i := \#\Delta_i$ . Assume that for only  $1 \leq j \leq s$  the divisor  $C_0 + \Delta_j$  is a linear chain. By the proof of Lemma 7.8, we have  $p_j \leq n_j / (n_j + 1)$  ( $j \leq s$ ).

If  $P_{\text{red}}.C_0 = C_0^2 + k \geq 2$ , then Proposition 7.10 (A) is true. So assume that  $C_0^2 \leq 1 - k \leq -3$ . Note that  $0 \leq P.C_0 = C_0^2 + \sum_{i=1}^k p_i \leq C_0^2 + (k - s) + \sum_{i=1}^s p_i \leq 1 - s + \sum_{i=1}^s p_i \leq 1 - \sum_{i=1}^s 1/(n_i + 1)$ . Suppose that  $\#\Delta_i = 1$  for  $1 \leq i \leq s_1$  and  $\#\Delta_i \geq 2$  for  $i \geq s_1 + 1$ . Then

$$0 \leq \sum_{i=s_1+1}^s 1/(n_i + 1) \leq 1 - s_1/2.$$

Note also that  $\#\Delta_j \geq 3$  for all  $s + 1 \leq j \leq k$ . Thus,

$$3k - s - s_1 = s_1 + 2(s - s_1) + 3(k - s) \leq \#P - 1 \leq 8.$$

These two highlighted inequalities imply that  $s = 2$  and  $(\#\Delta_1, \dots, \#\Delta_k) = (1, 1, 3, 3)$ .

Note that  $C_0$  meets the mid-component  $C_j$  of  $\Delta_j$  ( $j = 3, 4$ ). By the proof of Lemma 7.8, for every  $j$  with  $j \neq 0, 3, 4$ , we have  $p_j \leq 1/2$ . Thus  $0 \leq P.C_0 = C_0^2 + \sum_{i=1}^4 p_i \leq -3 + (1/2) + (1/2) + 1 + 1 = 0$ , so  $C_0^2 = -3$  and  $p_3 = p_4 = 1$ . Now  $0 \leq P.C_3 \leq C_3^2 + p_0 + (1/2) + (1/2)$  implies  $C_3^2 = -2$ . So  $P_{\text{red}}.(C_0 + C_3) = 2$  and Proposition 7.10 (A) is true.  $\square$

Now we assume that  $C_0$  meets exactly three components  $C_i$  ( $i = 1, 2, 3$ ) of  $\text{Supp}(P) - C_0$ . Let  $\Delta_i$  be the connected component of  $\text{Supp}(P) - C_0$  containing  $C_i$ . Set  $n_i := \#\Delta_i$ . Then  $\sum_{i=1}^3 n_i = \#P - 1 \leq 8$ . We may assume that  $n_1 \leq n_2 \leq n_3$ . Then  $n_3 \leq 6$  and  $n_1 \leq 2$ , so  $C_0 + \Delta_1$  is a linear chain. By the proof of Lemma 7.8, we have  $p_1 \leq n_1/(n_1 + 1) < 1$ . This and  $0 \leq P.C_0 = C_0^2 + p_1 + p_2 + p_3$ , together with  $C_0^2 \leq -2$ , imply that  $C_0^2 = -2$ . We shall apply Lemma 7.6 frequently, where  $G_0$  can be chosen as  $C_0$  or  $C_3$ .

**Lemma 7.13.** *Suppose that  $\#\Delta_i = 1$  for  $i = 1$  and 2 (this is true if  $\#\Delta_3 = 6$ ). Then Proposition 7.10 is true.*

*Proof.* By the proof of Lemma 7.8, we have  $p_i \leq 1/2$  for  $i = 1$  and 2. Now  $0 \leq P.C_0 = C_0^2 + p_1 + p_2 + p_3$  (and  $C_0^2 = -2$ ) imply  $p_3 = 1$  and  $p_i = 1/2$  ( $i = 1, 2$ ). By Lemma 7.11 and 7.12 (applied to  $C_3$ ), we may assume that  $C_3$  meets exactly three components  $C_0, C_4, C_5$  of  $\text{Supp}(P) - C_3$ . If  $C_3^2 = -2$ , then  $P_{\text{red}}.(C_0 + C_3) = 2$  and Proposition 7.10 (A) is true. Suppose that  $C_3^2 \leq -3$ . Then as above  $C_3^2 + p_0 + p_4 + p_5 = P.C_3 \geq 0$  implies that  $C_3^2 = -3$  and  $p_4 = p_5 = 1$ . (Of course,  $p_0 = 1$  is always assumed). Again by the same Lemmas we may assume that  $C_i$  ( $i = 4, 5$ ) meets exactly three components (one of which is  $C_3$ ). Then  $\#P \geq 10$ , a contradiction to the additional assumption  $\#P \leq 9$ .  $\square$

**Lemma 7.14.** *Suppose that  $C_0 + \Delta_i$  is a linear chain for all  $i = 1, 2, 3$ . Then Proposition 7.10 is true.*

*Proof.* Note that  $\sum_{i=1}^3 n_i = \#P - 1 \leq 8$ . By Lemma 7.13, we may assume that  $n_3 \leq 5$ . Except the cases below,  $P$  is negative (semi-) definite by Lemma 7.2 (which is impossible):

$$(n_1, n_2, n_3) = (1, 3, 4), (2, 2, 4), (2, 3, 3), (2, 2, 3).$$

In the first (resp. the last three) cases,  $\text{Supp}(P)$  supports a divisor  $D$  of type  $III^*$  (resp.  $IV^*$ ). We need to show that the coefficient  $\geq 2$  components of  $D$  are  $(-2)$ -curves and that  $P_{\text{red}} = L_{\text{red}}$  is given as in Proposition 7.10 (B1) in the first case. These follow from Lemma 7.6 applied to all  $0 \leq k \leq 8$ . For instance, in notation of Proposition 7.10 (B1), if we set  $-2 \geq G_k^2 = -x_k$  ( $k = 6, 8$ ) and

$G_j^2 = -2$  ( $j \neq 6, 8$ ), then  $\det(G_i \cdot G_j)_{0 \leq i, j \leq 8} = -4 + 3x_6 + 4x_8 - 2x_6x_8 > 0$  provided that  $G_k^2 \geq C_k^2$ ; also if  $C_i^2 = -2$  ( $i \neq 8$ ) then  $(-2)$ -components of  $P$  would support a  $C$  of type III\*.

When  $P_{\text{red}}$  is as in Proposition 7.10 (B1), one can check that the lattice  $\mathbf{Z}[K_X, C'_i s]$  generated by the divisor class of  $K_X$  and those of the nine curves in  $P$ , has determinant  $K_X^2 - 1$ . Note also that  $\rho(X) \leq 10 - K_X^2$ . So either  $K_X^2 = 0$  (and  $\kappa(X) = 1$ ),  $\text{Pic}(X) = \mathbf{Z}[K_X, C'_i s]$  (noting that  $\text{Pic}(X)$  is torsion free for  $\pi_1(X) = (1)$ ),  $\det(\text{Pic}(X)) = -1$  and Proposition 7.10 (B1) is true, or  $K_X^2 = 1$ ; but the latter situation implies, after a direction calculation, that  $K_X$  is numerically (and hence linearly, for  $\pi_1(X) = (1)$ ) equivalent to an effective integral divisor  $\sum k_i C_i$  with  $(k_0, k_1, \dots, k_8) = (10, 5, 7, 8, 4, 6, 1, 4, 2)$ , contradicting the assumption that  $p_g(X) = 0$ .  $\square$

**Lemma 7.15.** *Suppose that  $n_3 = \#\Delta_3 = 5$ . Then Proposition 7.10 is true.*

*Proof.* Since  $n_1 + n_2 = \#P - 1 - n_3 \leq 3$ , we have  $(n_1, n_2) = (1, 1), (1, 2)$  and  $C_0 + \Delta_i$  ( $i = 1, 2$ ) is a linear chain. By Lemma 7.13, we may assume that  $C_0 + \Delta_3$  is not a linear chain.

We shall apply Lemma 7.6 to deduce the result. The case  $\#P \leq 8$  can be reduced to the case  $\#P = 9$  because if an effective  $P_1$  with  $\#P_1 = 8$  supports a nef and big divisor then  $P$  with  $P > P_1$  supports a nef and big divisor too. So  $(n_1, n_2) = (1, 2)$ .

Suppose that  $\Delta_3$  is a linear chain. By Lemma 7.6, we have  $C_3^2 = -2$ , whence  $P_{\text{red}} \cdot (C_0 + C_3) = 2$  and Proposition 7.10 (A) is true. Indeed, if we set  $-2 \geq G_3^2 = -x_3$  and  $G_j^2 = -2$  ( $j \neq 3$ ) then  $0 < \det(G_{i,j})_{0 \leq i, j \leq 8}$  equals  $114 - 45x_3$  (when  $C_0$  meets the middle component of  $\Delta_3$ ), or  $98 - 40x_3$  (otherwise), provided that  $G_3^2 \geq C_3^2$ .

Suppose that  $\Delta_3$  is not a linear chain. Then it is of type  $I_0^*$  or  $D_5'$ . We denote by  $C_\ell$  the central component. Consider the case where  $\Delta_3$  is of type  $I_0^*$ . If  $C_3$  in  $\Delta_3$  (and meeting  $C_0$ ) is a tip component (resp. the central component  $C_\ell$ ) of  $\Delta_3$ , then applying Lemma 7.6, we have  $C_\ell^2 = -2$  (resp.  $C_\ell^2 = -2, -3$ ). Thus  $L \cdot C_\ell \geq 2$  and Proposition 7.10 (A) is true.

Consider the case where  $\Delta_3$  is of type  $D_5'$  so that  $C_\alpha + C_\beta + C_\ell$  is the ordered linear chain in  $\Delta_3$ . If  $C_3$  is  $C_\alpha$  (resp.  $C_\beta$ , or  $C_\ell$ , or a tip component  $C_\gamma \neq C_\alpha$  of  $\Delta_3$ ), applying Lemma 7.6, we have  $C_i^2 = -2$  for all  $C_i$  in  $C$  so that  $P_{\text{red}} \cdot C = 2$  and hence Proposition 7.10 (A) is true, where  $C$  equals  $C_0 + C_\alpha + C_\beta + C_\ell$  (resp.  $C_0 + C_\beta$ , or  $C_\ell$ , or  $C_0 + C_\gamma + C_\ell$ ).  $\square$

**Lemma 7.16.** *Suppose that  $n_3 = \#\Delta_3 = 4$ . Then Proposition 7.10 is true.*

*Proof.* As in the previous lemma, we only need to consider the case  $\#P = 9$ . Then  $n_1 + n_2 = \#P - 1 - n_3 = 4$  and  $(n_1, n_2) = (1, 3), (2, 2)$ . So  $C_0 + \Delta_1$  is a linear chain.

Consider the case that  $C_0 + \Delta_2$  is not a linear chain. Then  $(n_1, n_2) = (1, 3)$ . If  $C_2^2 = -2$ , then  $P_{\text{red}} \cdot (C_0 + C_2) = 2$  and Proposition 7.10 (A) is true. Suppose that  $C_2^2 \leq -3$ . By Lemma 7.8,  $\Delta_1 + \Delta_2 \leq \text{Supp}(N)$ , and by Lemma 7.7 with  $G_1^2 = -2$  (resp.  $G_2^2 = -3$ ) we have  $p_1 \leq 1/2$  (resp.  $p_2 \leq 1/2$ ), and the other two components of  $\Delta_2$  have coefficients less than or equal to  $1/4$  in  $P$ . This and  $0 \leq P \cdot C_0 = C_0^2 + p_1 + p_2 + p_3$  imply that  $p_3 = 1$ . By Lemma 7.11 and 7.12 we may assume that  $C_3$  meets exactly three components (one of which is  $C_0$ ), so  $\Delta_3$  is a linear chain. If  $C_3^2 = -2$ , then  $P_{\text{red}} \cdot (C_0 + C_3) = 2$  and Proposition 7.10

(A) is true. If  $C_3^2 \leq -3$ , then  $\Delta_3 \leq \text{Supp}(N)$  by Lemma 7.8; applying Lemma 7.7 with  $G_3^2 = -3$ , we have  $p_3 \leq 6/11$  (the coefficients of components of  $\Delta_3$  in  $P$  are respectively less than or equal to  $2/11, 4/11, 6/11, 3/11$ ); this leads to that  $P.C_0 < 0$ , a contradiction.

Therefore, we may assume that  $C_0 + \Delta_2$  is a linear chain but  $C_0 + \Delta_3$  is not a linear chain (see Lemma 7.14). If  $\Delta_3$  is a linear chain and  $C_3^2 = -2$ , then  $P_{\text{red}}.(C_0 + C_3) = 2$  and Proposition 7.10 (A) is true. If  $\Delta_3$  is a linear chain and  $C_3^2 \leq -3$ , then as above we have  $p_3 \leq 6/11$  and  $p_1 + p_2 \leq \sum_{i=1}^2 \frac{n_i}{n_i+1} \leq 4/3$ . This leads to that  $0 \leq P.C_0 = C_0^2 + p_1 + p_2 + p_3 \leq -2 + (4/3) + (6/11) < 0$ , a contradiction.

Thus we may assume that  $\Delta_3$  is not a linear chain, hence of type  $D_4'$  with the central component  $C_\ell$ . For both cases of  $(n_1, n_2) = (1, 3)$  and  $(2, 2)$ , if  $C_3$  is a tip component (resp.  $C_\ell$ ) of  $\Delta_3$ , then applying Lemma 7.6 we have  $C_i^2 = -2$  for all  $C_i$  in  $C$  so that  $P_{\text{red}}.C = 2$  and Proposition 7.10 (A) is true, where  $C$  equals  $C_0 + C_3 + C_\ell$  (resp.  $C_\ell$ ). This proves the lemma.  $\square$

**Lemma 7.17.** *Suppose that  $n_3 = \#\Delta_3 \leq 3$ . Then Proposition 7.10 is true.*

*Proof.* As in the previous lemmas, we may assume that  $\#P = 9$ , so  $(n_1, n_2, n_3) = (2, 3, 3)$ . Hence  $C_0 + \Delta_1$  is a linear chain. By Lemma 7.14, we may assume that  $C_0 + \Delta_3$  is not a linear chain. When  $C_0 + \Delta_i$  ( $i = 1$  or  $2$ ) is not a linear chain and  $C_i^2 = -2$ , we have  $P_{\text{red}}.(C_0 + C_i) = 2$  and Proposition 7.10 (A) is true. So assume that  $C_3^2 \leq -3$ . Then  $p_3 \leq 1/2$  by Lemmas 7.8 and 7.7 with  $G_3^2 := -3$ . Also we may assume either  $C_0 + \Delta_2$  is a linear chain or otherwise and  $C_2^2 \leq -3$  (and hence  $p_2 \leq 1/2$ ). If the former case occurs, by the proof of Lemma 7.8, we have  $p_i \leq n_i/(n_i+1)$  ( $i = 1, 2$ ) and  $0 \leq P.C_0 = C_0^2 + p_1 + p_2 + p_3 \leq -2 + (2/3) + (3/4) + (1/2) < 0$ , a contradiction. If the latter case occurs, then  $0 \leq P.C_0 \leq -2 + (2/3) + (1/2) + (1/2) < 0$ , a contradiction. This proves the lemma. The proof of Proposition 7.10 is also completed.  $\square$

## 8. SURFACES OF KODAIRA DIMENSION 1 OR 2

In this section we shall prove the two theorems below:

**Theorem 8.1.** *Let  $X$  be a minimal nonsingular projective surface of Kodaira dimension 2. Let  $L$  be a nef and big effective  $\mathbf{Q}$ -divisor. Then  $H^0(X, K_X + 3L_{\text{red}}) \neq 0$ .*

**Theorem 8.2.** *Let  $X$  be a minimal nonsingular projective surface of Kodaira dimension 1. Let  $L$  be a nef and big effective  $\mathbf{Q}$ -divisor.*

- (1) *We have  $H^0(X, K_X + 4L_{\text{red}}) \neq 0$ .*
- (2) *Suppose that  $H^0(X, K_X + 3L_{\text{red}}) = 0$ . Then  $L_{\text{red}}$  contains at least its name sake with 9 components given in Proposition 7.10 (B1). Further,  $\pi_1(X) = (1)$ ,  $\rho(X) = 10$ ,  $\det(\text{Pic}(X)) = -1$  and the elliptic fibration  $\pi : X \rightarrow \mathbf{P}^1$  has exactly two multiple fibres, and their multiplicities are 2 and 3. The  $\text{Pic}(X)$  is generated by the divisor class of  $K_X$  and those of the 9 components of  $L$ .*

We now prove Theorems 8.1 and 8.2 simultaneously. By Theorem 3.1, we may assume that  $q(X) = 0$ . We may also assume that  $H^0(X, K_X + L_{\text{red}}) = 0$ , so  $p_g(X) = 0$  and  $\chi(\mathcal{O}_X) = 1$ . By Proposition 2.4, the  $L_{\text{red}}$  is a connected rational tree and  $\pi_1(X) = (1)$ . So we can apply Proposition 7.10.

Consider first the case  $\#L \leq 9$  (this is true if  $\kappa(X) = 2$  by Proposition 7.10). We apply Proposition 7.10. If Proposition 7.10 (A) occurs, applying the Serre duality and Riemann Roch theorem, we have  $h^0(X, K_X + L_{\text{red}} + C) \geq \frac{1}{2}(K_X + L_{\text{red}} + C).(L_{\text{red}} + C) + \chi(\mathcal{O}_X) = \frac{1}{2}\{(K_X + L_{\text{red}}).L_{\text{red}} + (C^2 + K_X.C) + 2C.L_{\text{red}}\} + 1 \geq \frac{1}{2}\{(-2) + (-2) + 2 \times 2\} + 1 = 1$ , where the terms  $(-2)$  are due to the fact that both  $L_{\text{red}}$  and  $C$  are connected rational trees. Since  $2L_{\text{red}} \geq L_{\text{red}} + C$ , the theorems follow in this case.

Suppose Proposition 7.10 (B) occurs. As above we have  $h^0(X, K_X + C) \geq \frac{1}{2}(K_X + C).C + \chi(\mathcal{O}_X) = 0 + 1 = 1$ . If  $C$  is of type III\*, then  $L_{\text{red}}$  is given in Proposition 7.10 (B1) (so  $\kappa(X) = 1$ ) and we have  $4L_{\text{red}} \geq C$ ; thus both Theorems 8.2 and 8.1 are true by Lemma 8.3 below. If  $C$  is of other type, then  $3L_{\text{red}} \geq C$ . This proves the theorems.

It remains to consider the case where  $\#L \geq 10$ . So  $\kappa(X) = 1$  and  $\rho(X) = 10$  by Proposition 7.10. By Lemma 7.4 and the calculation above, we may proceed with the additional assumption that no divisor of elliptic fibre type is supported by some  $(-2)$ -components of  $\text{Supp}(L)$ . By Lemma 7.3, we have  $\rho(X) = 10$  and we may assume that  $\text{Pic}(X) \otimes \mathbf{Q}$  is generated by  $C_i$  ( $1 \leq i \leq 10$ ) in  $L_{\text{red}}$  after relabelling: first find 9 components of  $L_{\text{red}}$  having a negative definite intersection matrix, and then the 10th generator can be found from  $\text{Supp}(L)$  because  $L$  is nef and big (so not negative definite).

Theorefore,  $K_X$  is numerically equivalent to a  $\mathbf{Q}$ -linear combination of  $C_i$  ( $1 \leq i \leq 10$ ). Split the combination as  $L_2 - L_1$  so that  $K_X + L_1 \sim_{\mathbf{Q}} L_2$ , where both  $L_j$  are effective,  $(L_j)_{\text{red}} \leq \sum_{i=1}^{10} C_i$  and there is no common component of  $L_1$  and  $L_2$ . Since  $\kappa(X) = 1$ , we have  $L_2 > 0$ . Also  $L_2$  is nef, noting that  $K_X$  is nef.

Suppose that  $L_2$  is not big. Then  $0 = L_2^2 = L_2.K_X + L_2.L_1 \geq 0 + 0$ . Thus  $K_X.L_2 = 0$  and hence  $L_2$  is contained in fibres of the elliptic fibration  $\pi : X \rightarrow \mathbf{P}^1$ , noting that  $q(X) = 0$ , (so that  $K_X$  is numerically equal to a positive multiple of a fibre). This and the fact that  $L_2^2 = 0$  and fibre components are semi negative definite [Re], imply that  $L_2 = \sum b_j F_j$  where  $b_j$ 's are positive rational numbers and  $F_j$ 's are full fibres, whence  $(-2)$ -components of  $L_{\text{red}}$  ( $\geq (L_2)_{\text{red}}$ ) support an elliptic fibre, contradicting the additional assumption.

Therefore,  $L_2$  is nef and big. Thus  $L_1 \neq 0$  because  $K_X \sim_{\mathbf{Q}} L_2 - L_1$  is nef but not big. This and the fact that  $\#L_1 + \#L_2 = \#(L_1 + L_2) \leq 10$  imply that  $\#L_2 \leq 9$ .

So we are reduced to the case  $\#L \leq 9$  after replacing  $L_{\text{red}}$  by its subdivisor  $(L_2)_{\text{red}}$ . This proves the theorem.

**Lemma 8.3.** *Suppose that  $X$  is a minimal nonsingular projective surface of Kodaira dimension  $\kappa(X) = 1$  and  $p_g(X) = 0$ . Let  $D$  be the reduced divisor given in Proposition 7.10 (B1) (denoted as  $L_{\text{red}}$  there). Then the elliptic fibration  $\pi : X \rightarrow \mathbf{P}^1$  has exactly two multiple fibres, and their multiplicities are 2 and 3.*

*Proof.* We change the labelling and write  $D = \sum_{i=0}^8 D_i$ , where  $D_0$  meets  $D_1, D_5$  and  $D_8$ ;  $D_1 + \dots + D_4$  and  $D_5 + D_6 + D_7$  are linear chains;  $D_7^2 = -3$  and  $D_i^2 = -2$  ( $i \neq 7$ ). We can check that  $D$  supports a nef and big divisor (the Zariski positive part of  $D$ ):  $P = D_0 + \frac{1}{5}(D_4 + 2D_3 + 3D_2 + 4D_1) + \frac{1}{7}(D_7 + 3D_6 + 5D_5) + \frac{1}{2}D_8$ . Indeed,  $P^2 = P.D_0 = 1/70$ . By [No, Cor. 2.3] or the proof of Lemma 2.3, we have  $\pi_1(X) = (1)$ , whence  $q(X) = 0$  and  $\chi(\mathcal{O}_X) = 1$ . As in the proof of Lemma 7.4, we have  $\rho(X) \leq 10$ . We can check that the lattice  $\mathbf{Z}[K_X, D_i's]$  generated by the divisor

classes of  $K_X$  and those of the 9 curves of  $D$  has determinant  $-1$ . So this lattice equals  $Pic(X)$  and  $\rho(X) = 10$ , noting that  $Pic(X)$  is torsion free for  $\pi_1(X) = (1)$ .

By Lemma 2.5 (and the notation there) and by the canonical divisor formula we have  $K_X \sim_{\mathbf{Q}} (1 - \frac{1}{m_1} - \frac{1}{m_2})F_1$ . We still have to show that  $(m_1, m_2) = (2, 3)$ . Let  $F_3$  be the fibre of  $\pi$  containing the eight  $(-2)$ -components of  $D$ . Then  $F_3$  must be of type  $II^*$ , so there is a  $(-2)$ -curve  $G$  such that  $G$  and the eight  $(-2)$ -components of  $D$  support the fibre  $F_3$  (whence  $G.D_4 = 1$  and  $G.D_i = 0$  ( $i \neq 4, 7$ )).

On the other hand, express  $G \sim kK_X + \sum_{i=0}^8 d_i D_i$  for some integers  $k, d_i$ . Intersecting the equality by  $K_X$ , we obtain  $0 = d_7 D_7.K_X = d_7$ . So  $kK_X \sim G - \sum_{i \neq 7} d_i D_i$  and the RHS is supported on the fibre  $F_3$  and has self intersection 0 (because  $K_X^2 = 0$ ). Since the fibre components are negative semi-definite, this implies that the RHS is a multiple of  $F_3$ . Now  $G$  has coefficient 1 in  $F_3$ , so the RHS  $= F_3$ . Namely,  $kK_X \sim F_3$ , or  $K_X \sim_{\mathbf{Q}} F_3/k$ . Comparing with the expression of  $K_X$  in the previous paragraph, we obtain:  $\frac{1}{k} = (1 - \frac{1}{m_1} - \frac{1}{m_2})$ . Simplifying, we obtain:  $m_1 m_2 = k(m_1 m_2 - m_1 - m_2)$ . Since  $m_1$  and  $m_2$  are coprime, we have  $m_1 m_2 | k$ . So  $k = m_1 m_2$  and  $m_1 m_2 - m_1 - m_2 = 1$ , or  $1 = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_1 m_2}$ . One sees then  $(m_1, m_2) = (2, 3)$ . By the way, then  $F_3 \sim F_1 \sim 6K_X$ . Intersecting this relation with  $D_7$ , we see that  $D_7$  is a 6-section and  $D_7.G = 4$ . This proves the lemma.  $\square$

*Remark 8.4.* The non-vanishing of  $H^0(X, K_X + L_{\text{red}})$  or  $H^0(X, K_X + \lceil L \rceil)$ , when  $\kappa(X) = 1$ , is subtle and is not easy to be proven at all. Indeed, suppose that  $X$  is a minimal nonsingular projective surface with Kodaira dimension 1,  $q(X) = 0$  and  $p_g(X) = 0$ . Let  $\pi : X \rightarrow \mathbf{P}^1$  be the elliptic fibration. Suppose that there is a type  $II^*$  elliptic fibre  $F_0$  and also there is a 6-section  $E (\cong \mathbf{P}^1)$  such that  $E$  meets the multiplicity-6 component of  $F_0$ . (We have this possible situation in mind:  $\pi$  has exactly two multiple fibres. Their multiplicities are 2, 3; see Lemma 2.5). Then  $L = \frac{1}{6n}(E + nF_0)$  is nef and big for  $n \gg 0$ . Clearly,  $L_{\text{red}}$  is a connected rational tree (hence also of simple normal crossing) and the round up  $\lceil L \rceil = L_{\text{red}}$ . By the Kawamata-Viehweg vanishing and Riemann-Roch theorem, we have  $h^0(X, K_X + L_{\text{red}}) = \frac{1}{2}(K_X + L_{\text{red}}).L_{\text{red}} + \chi(\mathcal{O}_X) = (-1) + 1 = 0$ . (However, as in the proof of Theorem 8.2 or Lemma 7.4, we have  $H^0(X, K_X + 2L_{\text{red}}) \neq 0$ .) Therefore, to prove the desired non-vanishing, one has to show that the above geometric situation will never occur.

## 9. ON NON-MINIMAL SURFACES

Let's fix some notation first. Let  $X$  be a surface and  $L$  is nef and big effective  $\mathbf{Q}$ -divisor such that  $K_X + L$  is nef. Let  $M := \lceil L \rceil$ . It's clear that  $H^i(X, K_X + M) = 0$  for  $i = 1, 2$ . Let  $C$  be an irreducible component of  $L$ , then we denote by  $v_C(L)$  the coefficient of  $L$  along  $C$ .

We assume that  $X$  is non-minimal with  $p_g = q = 0$  and  $\kappa = -\infty$ . There exists at least a  $(-1)$  curve.

**Lemma 9.1.** *Let  $X$  be a non-singular surface with  $p_g = q = 0$ . Let  $M$  be an effective divisor on  $X$ . The following are equivalent.*

- (1)  $H^i(X, K_X + M) = 0$  for  $i = 0, 1$ .
- (2) *there is a sequence of divisors  $0 = M_0 < \dots < M_n = M$  such that  $M_i = M_{i-1} + C_i$ ,  $C_i.M_{i-1} = 1$  and  $H^j(K_X + M_i) = 0$  for  $j = 0, 1, 2$ . Each  $C_i$  is a rational curve.*

- (3) there is a sequence of divisors  $0 = M_0 < \dots < M_n = M$  as above such that  $M_{i_0} = M_{\text{red}}$  for some  $i_0$ .

A sequence in (3) is called a NN (non-nef) sequence of  $M$ .

*Proof.* It's clear that (2) implies (1) by direct computation on cohomology. To see that (1) implies (2), we recall Riemann-Roch theorem,

$$0 = h^0(X, M) = \chi(X, M) = 1 + \frac{1}{2}(K_X + M).M.$$

Hence we may assume that  $K_X + M$  is non-nef. Pick an irreducible component  $C$  of  $M$  such that  $(K_X + M).C < 0$ . Let  $M_n := M, M_{n-1} = M - C$ . We consider now the exact sequence:

$$0 \rightarrow \mathcal{O}(K_X + M_{n-1}) \rightarrow \mathcal{O}(K_X + M_n) \rightarrow \mathcal{O}_C(d) \rightarrow 0,$$

where  $d = (K_X + M).C < 0$ . We have  $H^i(X, K_X + M_{n-1}) = 0$  for  $i = 0, 1, 2$ . Moreover,  $d = -1$ . Proceed this process inductively, we are done.

Suppose now that we have a sequence as in (2). Let  $j_0$  be the smallest number that  $M_{j_0} > M_{\text{red}}$ . Then clearly, there is a curve  $C$  such that  $v_C(L_{j_0}) = 1, v_C(L_{j_0-1}) = 0$ . That is,  $M_{j_0}$  is obtained from  $M_{j_0-1}$  by joining a curve  $C$  which is not in its support.

If  $M_{j_0}$  is reduced, then we are done. Suppose now that  $M_{j_0}$  is not reduced. Let  $k$  be the largest number such that  $M_k$  is obtained by joining a curve  $\Gamma$  which is in the support of  $M_{k-1}$ . We now have a sequence

$$M_{k-1} \xrightarrow{+\Gamma} M_k \xrightarrow{+C_{k+1}} M_{k+1} \xrightarrow{+C_{k+2}} \dots \rightarrow M_{j_0-1} \xrightarrow{+C} M_{j_0}.$$

Note that  $C_{k+1}$  is not in the support of  $M_k$  and  $M_k.C_{k+1} = 1$ , thus  $\Gamma.C_{k+1} = 0$ . Hence joining  $\Gamma$  commutes with joining  $C_{k+1}$ . Therefore, let  $M'_k := M_{k-1} + C_{k+1}$ , we have

$$M_{k-1} \xrightarrow{+C_{k+1}} M'_k \xrightarrow{+\Gamma} M_{k+1} \xrightarrow{+C_{k+2}} \dots \rightarrow M_{j_0-1} \xrightarrow{+C} M_{j_0}.$$

Similarly, one can obtain a sequence

$$M_{k-1} \xrightarrow{+C_{k+1}} M'_k \xrightarrow{+C_{k+2}} M'_{k+1} \xrightarrow{+C_{k+3}} \dots \rightarrow M'_{j_0-1} \xrightarrow{+\Gamma} M_{j_0}.$$

Now  $L'_{j_0-1} > M_{\text{red}}$  with smaller non-reduced part. Inductively, we are done.  $\square$

We assume now that  $\kappa = -\infty$ . We are now going to verify the conjecture. Let  $f : X \rightarrow {}_1X$  be a contraction map with exceptional divisor  $E$ . And let  ${}_mX$  be a minimal model of  $X$ . We write  ${}_1L := f_*(L)$ . Then  $L = f^*{}_1L - aE$  for some  $a \geq 0$  if  $L$  is nef. And so is  $K_X + L$ . By direct computation, one sees that if  $L$  is nef and big with  $K_X + L$  nef on  $X$ , then  ${}_iL$  is nef and big and  $K_{{}_iX} + {}_iL$  is nef on  ${}_iX$  for  $0 \leq i \leq m$ . We assume that  ${}_mX = \mathbb{F}_d$  for some  $d \geq 0$ .

In order to handle non-minimal surface, we need the following filling-up lemma:

**Lemma 9.2.** *Let  $C$  be a reduced irreducible curve on  $X$ , with image  ${}_mC$  under a birational map  $f : X \rightarrow {}_mX$ . Given an effective divisor  $M$  of the form  $C + \sum e_i E_i$  which possesses a NN-sequence, where  $E_i$  are exceptional divisors, then  $f^*({}_mC)$  has a NN-sequence.*

*Proof.* We prove by induction on number of exceptional divisors. Let

$$f : X \xrightarrow{g} {}_1X \xrightarrow{h} {}_mX,$$

where  $g$  contracting one  $(-1)$  curve  $E$ .

By induction hypothesis, we may assume that there is a NN-sequence on  ${}_1X$

$${}_1M_0 := g(M_{\text{red}}) \rightarrow {}_1M_1 \rightarrow \dots \rightarrow {}_1M_k := h^*({}_mC).$$

We claim that one can produce a NN-sequence on  $X$  out of this.

First of all, we may assume that there is  ${}_1M_{j_0} = (h^*({}_mC))_{\text{red}}$ .

We next distinguish the following two cases

**Case 1.**  $E$  meets only one other component, say  $P$ . Let  $j_0 \leq j_1 \leq j_2 \leq \dots \leq j_r \leq k$  be those indexes such that  ${}_1M_{j_i}$  is obtained by joining  $P$  for  $i = 1, \dots, r$ .

Let  $M_{j_0} := (f^*({}_mC))_{\text{red}}$  which clearly possesses a NN-sequence. For  $j_0 < i < j_1$ , we let  $M_i := g^*({}_1M_i)$  which clearly satisfies the requirement because we are joining curves which are unchanged under  $g$ . Let  $M_{j_1} := M_{j_1-1} + P$ ,  $N_{j_1} := M_{j_1} + E$ . Then  $N_{j_1} = g^*{}_1M_{j_1}$ . Then for  $j_1 < i < j_2$ , we let  $M_i := g^*({}_1M_i)$ . Inductively, we have a sequence

$$\begin{aligned} M_0 = M_{\text{red}} \rightarrow \dots \rightarrow M_{j_0} \rightarrow \dots \rightarrow M_{j_1} \xrightarrow{+E} N_{j_1} \rightarrow M_{j_1+1} \rightarrow \dots \\ \rightarrow M_{j_2} \rightarrow N_{j_2} \rightarrow \dots \rightarrow N_{j_r} \rightarrow \dots \rightarrow M_k. \end{aligned}$$

It's easy to verify that this is a NN-sequence for  $M_k = f^*({}_mC)$ .

**Case 2.**  $E$  meets two other components, say  $P_1, P_2$ . Let  $j_0 \leq j_1 \leq j_2 \leq \dots \leq j_r \leq k$  be those indexes such that  ${}_1M_{j_i}$  is obtained by joining  $P_1$  or  $P_2$  for  $i = 1, \dots, r$ . It's easy to see that similar construction holds.  $\square$

We are now ready to prove the main theorem.

*Proof. Step 1.* Suppose that the support of  ${}_mL$  has only one horizontal curve, say  $C$ . We may write  $L = aC + \sum_p b_p B_p + \Delta_p$ , where  $B_p$  are the curves that meets  $C$ ,  $\Delta_p$  are the remaining part that supports on the fiber  $F_p$  over  $p$ .

Along each reducible fibers  $F_p$ , after contractions, it turns out to be of the type  $E_1 + E_2$  with  $E_i$  being  $(-1)$ -curves. In fact, by changing minimal models, we may assume that  ${}_{i+1}F_p$  is obtained by blowing-up along a point not at  $C \cap ({}_iF_p)$ , and  $C^2 = -d$ .

Since  $(K_X + L).F = a - 2 \geq 0$ , we have  $a \geq 2$ . Moreover, if  $d \geq 2$ , then we have  $\sum b_p \geq ad \geq 2d \geq d + 2$  for  $L.C \geq 0$ . If  $d = 1$ , then  $(K_X + L).C \geq 0$  implies that  $\sum b_p \geq a + 1 \geq 3 = d + 2$ . It follows that  $\sum [b_p] \geq d + 2$ .

Suppose on the contrary that  $H^0(X, K_X + M) = 0$ , then  $M$  has a NN-sequence. Where

$$M = [a]C + \sum_p [b_p]B_p + [\Delta_p].$$

We claim that there exist a NN-sequence for  $M' := [a]C + \sum_p [b_p]F_p$  by *filling-up fibers*. Grant this claim, then  $H^0(X, K_X + M') = 0$ . However,

$$K_X + M' = K_{X/{}_mX} + M' - 2C - (d + 2)F \geq 0$$

is effective. This leads to a contradiction.  $\square$

**Lemma 9.3.** *Let  $E$  be an irreducible component of  $M$  which is a  $(-1)$  curve that meet more than three other components. Then  $v_E(M) = 1$ .*

*Proof.* Suppose on the contrary that  $v_E(M) \geq 2$ . Then there is  $L_k$  in the NN sequence such that  $v_E(L_j) = 1$  for  $j_0 \leq j \leq k - 1$  and  $v_E(L_k) = 2$ . However,  $1 = L_{k-1}.C_k \geq 3 - 1$  which is absurd.  $\square$

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