THE CANONICAL VOLUME OF 3-FOLDS OF GENERAL TYPE WITH $\chi \leq 0$

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Abstract. We prove that the canonical volume $K^3 \geq \frac{1}{30}$ for all 3-folds of general type with $\chi(\mathcal{O}) \leq 0$. This bound is sharp.

1. Introduction

Let V be a nonsingular projective 3-fold of general type. According to Mori's Minimal Model Program (see for instance [15, 16, 19]), V has at least one minimal model X which is normal projective with at worst $\mathbb{Q}$-factorial terminal singularities. Denote by $K^3 := K_X^3$. Since it is uniquely determined by the birational equivalence class of V, K^3 is usually referred to as *the canonical volume of* V, also written as $\mathrm{Vol}(V)$. In the study of 3-folds of general type, a major difficulty arises when K^3 is only a small rational number, rather than an integer. For example, among known ones by Fletcher-Reid (cf. [10], p151), $\mathrm{Vol}(V)$ could be as small as $\frac{1}{420}$. It is a fact that the birational invariant $\mathrm{Vol}(V)$ strongly affects the geometry of V. So a natural and interesting question is to find the sharp lower bound v_3 of K^3 among all those nonsingular 3-folds V of general type.

There have been some relevant known results already:

- There exists a constant $v_3 > 0$ such that $\mathrm{Vol}(V) \geq v_3$ for all threefolds of general type. This is proved by Hacon and M^{c}kernan [12], Takayama [21] and Tsuji [22];
- It is proved by the second author [5] that $\mathrm{Vol}(V) \geq \frac{1}{3}$ for all 3-folds of general type with $p_g(V) := \dim H^3(V, \mathcal{O}_V) \geq 2$ and the bound "$\frac{1}{3}$" is sharp.

In this paper we would like to prove the following:

Theorem 1.1. *Let V be a nonsingular projective 3-fold of general type with $\chi(\mathcal{O}_V) \leq 0$. Then*

(i) $\mathrm{Vol}(V) \geq \frac{1}{30}$.
(ii) *When* $\mathrm{Vol}(V) = \frac{1}{30}$, *$V$ has the invariants:* $p_g(V) = 1$, $q(V) = 0$, $\chi(\mathcal{O}_V) = 0$, $P_2(V) = 1$, $P_3(V) = 2$, $P_4(V) = 3$ *and* $P_5(V) =$

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4. *Furthermore any minimal model of V has exactly 3 virtue baskets of singularities (in the sense of Reid):* $1 \times \frac{1}{2}(1,-1,1)$, $1 \times \frac{1}{3}(1,-1,1)$, $1 \times \frac{1}{5}(1,-1,1)$.

The next example shows that the lower bound of $\mathrm{Vol}(V)$ in Theorem 1.1(i) is optimal.

Example 1.2. (cf. [10], p151) The canonical hypersurface $X_{28} \subset \mathbb{P}(1,3,4,5,14)$ has the canonical volume $K^3 = \frac{1}{30}$, $p_g = 1$, $q = 0$, $\chi(\mathcal{O}_{X_{28}}) = 1$. X_{28} has 3 terminal singularities: $1 \times \frac{1}{2}(1,-1,1)$, $1 \times \frac{1}{3}(1,-1,1)$, $1 \times \frac{1}{5}(1,-1,1)$.

This note also contains some effective results. For example we will prove the following:

Corollary 1.3. *Let V be a nonsingular projective 3-fold of general type with $q := h^1(\mathcal{O}_V) > 0$. Then* $\mathrm{Vol}(V) \geq \frac{1}{22}$.

The paper is organized as the following. In section 2, we study the pluricanonical maps. We obtained, in Theorem 2.5, a lower bound $> \frac{1}{30}$ when the plurigenera are large. In section 3, we consider irregular threefolds. Combining results obtained in these two sections, the only unknown case has the information: $\chi(\mathcal{O}_X) = 0$, $q(X) = 0$, $p_g(X) = 1$ and $P_5(X) < 5$. Thus in sections 4 we classify all possible types of singularities and hence are able to complete the proof of the main theorem.

Throughout the paper $\sim$ means linear equivalence while $\equiv$ denotes the numerical one.

2. Bounding K^3 via φ_m

In order to get an effective lower bound of K^3 we need to study the m-canonical map φ_m. Let X be a minimal projective 3-fold of general type (admitting at worst $\mathbb{Q}$-factorial terminal singularities) with

$$P_{m_0} = P_{m_0}(X) := \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(m_0K_X)) \geq 2$$

for some integer $m_0 > 0$, where K_X is a canonical divisor of X.

2.1. Set up for φ_{m_0}. We study the m_0-canonical map $\varphi_{m_0} : X \dashrightarrow \mathbb{P}^{P_{m_0}-1}$ which is only a rational map. First of all we fix an effective Weil divisor $K_{m_0} \sim m_0K_X$. By Hironaka's big theorem, we can take successive blow-ups $\pi : X' \to X$ such that:

(i) X' is smooth;
(ii) the movable part of $|m_0K_{X'}|$ is base point free;
(iii) the support of $\pi^*(K_{m_0})$ is of simple normal crossings.

Set $g_{m_0} := \varphi_{m_0} \circ \pi$. Then g_{m_0} is a morphism by assumption. Let $X' \xrightarrow{f} B \xrightarrow{s} W'$ be the Stein factorization of g_{m_0} with W' the image of X' through g_{m_0}. In summary, we have the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\ f\ } & B \\ {\scriptstyle\pi}\downarrow & \searrow^{g_{m_0}} & \downarrow{\scriptstyle s} \\ X & \overset{\varphi_{m_0}}{\dashrightarrow} & W' \end{array}$$

We recall the definition of $\pi^*(K_X)$ and denote by $r(X)$ the Cartier index of X. Then $r(X)K_{X'} = \pi^*(r(X)K_X) + E_\pi$ where E_π is a sum of exceptional divisors. One defines $\pi^*(K_X) := K_{X'} - \frac{1}{r(X)}E_\pi$. So, whenever we take the round up of $m\pi^*(K_X)$, we always have $\lceil m\pi^*(K_X)\rceil \le mK_{X'}$ for any integer $m > 0$. We may write $m_0K_{X'} =_{\mathbb{Q}} \pi^*(m_0K_X) + E_{m_0} = M_{m_0} + Z_{m_0}$, where M_{m_0} is the movable part of $|m_0K_{X'}|$, Z_{m_0} the fixed part and E_{m_0} an effective $\mathbb{Q}$-divisor which is a $\mathbb{Q}$-sum of distinct exceptional divisors. We may also write $m_0\pi^*(K_X) =_{\mathbb{Q}} M_{m_0} + E'_{m_0}$, where $E'_{m_0} = Z_{m_0} - E_{m_0}$ is an effective $\mathbb{Q}$-divisor.

If $\dim(B) \ge 2$, a general member S of $|M_{m_0}|$ is a nonsingular projective surface of general type by Bertini's theorem and by the easy addition formula for Kodaira dimension.

If $\dim(B) = 1$, a general fiber S of f is an irreducible smooth projective surface of general type, still by the easy addition formula for Kodaira dimension. We may write

$$M_{m_0} = \sum_{i=1}^{a_{m_0}} S_i \equiv a_{m_0}S$$

where the S_i is a smooth fiber of f for all i and $a_{m_0} \ge P_{m_0}(X) - 1$.

In both cases we call S *a generic irreducible element of* $|M_{m_0}|$. Denote by $\sigma : S \longrightarrow S_0$ the blow-down onto the smooth minimal model S_0.

2.2. Assumptions. We need some assumptions to estimate K^3.

(1) Keep the same notations as above, we define

$$p = \begin{cases} 1 & \text{if } \dim(B) \ge 2 \\ a_{m_0} & \text{if } \dim(B) = 1. \end{cases}$$

(2) Take a generic irreducible element S of $|M_{m_0}|$. Assume that $|G|$ is a movable complete linear system on S. Also assume that a generic irreducible element C of $|G|$ is smooth.

(3) Assume there is a rational number $\beta > 0$ such that $\pi^*(K_X)|_S - \beta C$ is numerically equivalent to an effective $\mathbb{Q}$-divisor on S.

Set $\alpha = (m - 1 - \frac{m_0}{p} - \frac{1}{\beta})\xi$ and $\alpha_0 := \lceil\alpha\rceil$.

Under Assumptions 2.2, one has

$$K^3 \ge \frac{p}{m_0}\pi^*(K_X)^2 \cdot S \ge \frac{p\beta}{m_0}(\pi^*(K_X) \cdot C). \tag{2.1}$$

So it suffices to estimate the rational number $\xi := (\pi^*(K_X)\cdot C)_{X'}$.

We need the following theorem to study the lower bound of ξ:

Theorem 2.3. *Let $m>0$ be an integer. Under Assumptions 2.2, the inequality*

$$\xi \geq \frac{\deg(K_C)+\alpha_0}{m}$$

holds if one of the following conditions is satisfied:

(i) $\alpha_0 \geq 2$;
(ii) $\alpha > 0$ *and C is an even divisor on S.*

Proof. We consider the sub-linear system

$$|K_{X'} + \lceil (m-1)\pi^*(K_X) - \frac{1}{p}E'_{m_0}\rceil| \subset |mK_{X'}|.$$

Take a generic irreducible element S of $|M_{m_0}|$. Noting that $(m-1)\pi^*(K_X) - \frac{1}{p}E'_{m_0} - S \equiv (m-1-\frac{m_0}{p})\pi^*(K_X)$ is nef and big whenever $\alpha>0$, the Kawamata-Viehweg vanishing theorem [23, 14] gives the surjective map

$$\begin{aligned} &H^0(X', K_{X'} + \lceil (m-1)\pi^*(K_X) - \frac{1}{p}E'_{m_0}\rceil) \\ &\longrightarrow H^0(S, K_S + \lceil (m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0}\rceil|_S). \end{aligned} \tag{2.2}$$

Now consider a generic irreducible element $C\in|G|$. By assumption there is an effective $\mathbb{Q}$-divisor H on S such that

$$\frac{1}{\beta}\pi^*(K_X)|_S \equiv C + H.$$

By the vanishing theorem again, whenever $m-1-\frac{m_0}{p}-\frac{1}{\beta}>0$ which yields that

$$((m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0})|_S - C - H \equiv (m-1-\frac{m_0}{p}-\frac{1}{\beta})\pi^*(K_X)|_S$$

is nef and big, we have the surjective map

$$\begin{aligned} &H^0(S, K_S + \lceil ((m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0})|_S - H\rceil) \\ &\longrightarrow H^0(C, K_C + D) \end{aligned} \tag{2.3}$$

where $D := \lceil ((m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0})|_S - C - H\rceil|_C$ is a divisor on C. Noting that C is nef on S, we have $\deg(D)\geq\alpha$ and thus $\deg(D)\geq\alpha_0$.

Whenever either $\deg(D)\geq 2$ or C is an even divisor and $m-1-\frac{m_0}{p}-\frac{1}{\beta}>0$ ($\deg(D)\geq 2$ automatically follows), $|K_C+D|$ is base point free by the curve theory. Denote by $|N_m|$ the movable part of

$|K_S+\lceil((m-1)\pi^*(K_X)-S-\frac{1}{p}E'_{m_0})|_S-H\rceil|$. Applying Lemma 2.7 of [7] to surjective maps (2.2) and (2.3), one has

$$m\pi^*(K_X)|_S\geq N_m \ \text{ and } \ (N_m\cdot C)_S\geq 2g(C)-2+\deg(D).$$

So $m\xi\geq \deg(K_C)+\alpha_0$. We are done. □

Remark 2.4. A technical problem in utilizing Theorem 2.3 is to verify Assumptions 2.2. To avoid unnecessary redundancy, we only copy several technical results here without proof. Note that the most complicated situation is the one with $\dim(B)=1$, in which case we set $b:=g(B)$, the geometric genus.

(1) Usually we will take $G\leq\sigma^*(K_{S_0})$ whenever $p_g(S)\geq 2$ or $G=2\sigma^*(K_{S_0})$, $4\sigma^*(K_{S_0})$ otherwise;

(2) When $g(B)>0$, it is proved in Lemma 3.4 of [8] that

$$\pi^*(K_X)|_S\sim\sigma^*(K_{S_0}).$$

So one may take $\beta=1$ or $\frac{1}{2}$ or $\frac{1}{2}$.

(3) When $g(B)=0$, it is proved in Lemma 3.3 of [8] that

$$\pi^*(K_X)|_S-\widetilde{\beta}_n\sigma^*(K_{S_0})$$

is numerically equivalent to an effective $\mathbb{Q}$-divisor for a sequence of positive rational numbers $\{\widetilde{\beta}_n\}$ with $\widetilde{\beta}_n\mapsto\frac{p}{m_0+p}$. So one may take $\beta=\widetilde{\beta}_n$ or $\frac{1}{2}\widetilde{\beta}_n$ or $\frac{1}{4}\widetilde{\beta}_n$ accordingly.

(‡) Note that a special situation (with $m_0=1$) of this theory has already appeared in [5], as 2.8 and Lemma 3.4.

Now we are ready to estimate ξ and K^3.

Theorem 2.5. *Let V be a nonsingular projective 3-fold of general type. Then*

(i) $\mathrm{Vol}(V)\geq\frac{1}{22}$ *if* $P_4(V)\geq 5$;

(ii) $\mathrm{Vol}(V)\geq\frac{1}{25}$ *if* $P_5(V)\geq 5$, $p_g(V)>0$ *and* $\dim(B)\geq 2$.

(iii) $\mathrm{Vol}(V)\geq\frac{8}{45}$ *if* $P_5(V)\geq 5$, $p_g(V)>0$ *and* $\dim(B)=1$.

Proof. Take a minimal model X of V. We study $|mK|$ on X. Keep the same set up as in 2.1.

Part (i). We can take $m_0=4$. We will study according to the value of $\dim(B)$. Take a generic irreducible element S of $|M_4|$.

If $\dim(B)=3$, we know that $p=1$ by definition. In this case we know $S\sim M_5$ and that $|S|$ gives a generically finite morphism. Set $G:=S|_S$. Then $|G|$ is base point free and $\varphi_{|G|}$ gives a generically finite map. So a generic irreducible element C of $|G|$ is a smooth curve of genus ≥ 2. If $\varphi_{|G|}$ gives a birational map, then $\dim\varphi_{|G|}(C)=1$ for a general member C. The Riemann-Roch and Clifford's theorem on C says $C^2=G\cdot C\geq 2$. If $\varphi_{|G|}$ gives a generically finite map of degree ≥ 2, since $h^0(S,G)\geq h^0(X',S)-1\geq 4$, Lemma 2.2 of

[6] gives $C^2 \geq 2h^0(S,G) - 4 \geq 4$. Anyway we have $C^2 \geq 2$. So $\deg(K_C) = (K_S + C)C > 2C^2 \geq 4$. We see $\deg(K_C) \geq 6$ because it is even. One may take $\beta = \frac{1}{4}$ since $4\pi^*(K_X)|_S \geq C$. Now if we take a very big m such that $\alpha > 1$ then Theorem 2.3 gives:

$$m\xi \geq \deg(K_C) + (m - 1 - m_0 - \frac{1}{\beta})\xi.$$

This gives $\xi \geq \frac{2}{3}$. If we take $m = 11$. Then $\alpha = 2\xi > 1$. Theorem 2.3 says $\xi \geq \frac{8}{11}$. So inequality (2.1) gives $K^3 \geq \frac{1}{22}$.

If $\dim(B) = 2$, we know that $|G| := |S|_S|$ is composed with a pencil of curves. A generic irreducible element C of $|G|$ is a smooth curve of genus ≥ 2, so $\deg(K_C) \geq 2$. Furthermore we have $h^0(S,G) \geq h^0(X', S) - 1 \geq 4$. So $G \equiv \widetilde{a}C$ for $\widetilde{a} \geq h^0(S,G) - 1 \geq 3$. This means $4\pi^*(K_X)|_S \geq S|_S \geq_{\text{numerically}} 3C$. So we may take $\beta = \frac{3}{4}$. Now take a very big m. Theorem 2.3 gives $\xi \geq \frac{6}{19}$. Take $m = 10$. Then $\alpha \geq \frac{22}{19} > 1$. We get $\xi \geq \frac{2}{5}$. So inequality (2.1) gives $K^3 \geq \frac{3}{40} > \frac{1}{22}$.

If $\dim(B) = 1$ and $b = g(B) > 0$, there is an induced fibration $f : X' \longrightarrow B$. Recall that S is a general fiber of f and S can be a nonsingular surface of general type of any numerical type. One has $p = a_4 \geq P_4 \geq 5$ by the Riemann-Roch and Clifford's theorem. Set $G := 4\sigma^*(K_{S_0})$. Because $|4K_{S_0}|$ is base point free by Bombieri [2], $|G| = |4\sigma^*(K_{S_0})|$ is also base point free. Denote by C a generic irreducible element of $|G|$. Then C is smooth and $\deg(K_C) = (K_S + C)C \geq (\pi^*(K_X)|_S + C)C = (\sigma^*(K_{S_0}) + C)C = 20\sigma^*(K_{S_0})^2 \geq 20$. By Remark 2.4(2), we can take $\beta = \frac{1}{4}$ since $\pi^*(K_X)|_S \sim \frac{1}{4}(4\sigma^*(K_{S_0}))$. Now if we take a very big m, Theorem 2.3 gives $\xi \geq \frac{100}{29}$. Inequality (2.1) gives $K^3 \geq \frac{125}{116} > \frac{1}{22}$.

If $\dim(B) = 1$ and $b = g(B) = 0$, we have $p \geq P_5 - 1 \geq 4$. Take $m_0 = 4$. We study two cases separately: (a) $p_g(S) > 0$; (b) $p_g(S) = 0$.

First we consider case (a). We still set $G := 2\sigma^*(K_{S_0})$. Take a generic irreducible element C of $|G|$. By an established theorem (see Bombieri [2], Reider [20], Catanese-Ciliberto [3], and P. Francia [11] or directly refer to Theorem 3.1 in the survey article by Ciliberto [9]), $|2K_{S_0}|$ is always base point free. We get $\deg(K_C) = (K_S + C)C > C^2 \geq 4$. So actually $\deg(K_C) \geq 6$ due to its evenness. We only have to find a suitable β. Remark 2.4(3) says that one can find a sequence of positive rational numbers $\{\widetilde{\beta}_n\}$ with $\widetilde{\beta}_n \mapsto \frac{p}{p+4} \geq \frac{1}{2}$ such that $\pi^*(K_X)|_S - \widetilde{\beta}_n\sigma^*(K_{S_0})$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. Take $\beta_n := \frac{1}{2}\widetilde{\beta}_n$. Then $\pi^*(K_X)|_S - \beta_n C$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. We know $\beta_n \mapsto \frac{1}{4}$ whenever $p = 4$. When m is very big, Theorem 2.3 gives $\xi \geq 1$. So inequality (2.1) gives $K^3 \geq \frac{1}{4} > \frac{1}{22}$.

Finally we consider the case (b). We set $G := 4\sigma^*(K_{S_0})$. The surface theory tells us that $|G|$ is base point free and a generic irreducible element C of $|G|$ is a smooth curve. Because

$$\deg(K_C) = (K_S + C)C \geq (\pi^*(K_X)|_S + C)C > C^2 \geq 16,$$

again we see $\deg(K_C) \geq 18$. Similar to the case (a), we know that $\pi^*(K_X)|_S - \widetilde{\beta}_n\sigma^*(K_{S_0})$ is numerically equivalent to an effective $\mathbb{Q}$-divisor for a rational number sequence $\{\widetilde{\beta}_n\}$ with $\widetilde{\beta}_n \mapsto \frac{p}{p+4} \geq \frac{1}{2}$. Take $\beta_n := \frac{1}{4}\widetilde{\beta}_n$. Then $\pi^*(K_X)|_S - \beta_n C$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. We know $\beta_n \mapsto \frac{1}{8}$ whenever $p = 4$. When m is very big, Theorem 2.3 gives $\xi \geq \frac{9}{5}$. Take $m = 11$. Then $\alpha \geq \xi > 1$. One gets $\xi \geq \frac{20}{11} > \frac{9}{5}$. So inequality (2.1) gives $K^3 \geq \frac{5}{22} > \frac{1}{22}$.

Comparing what we have proved, we see $K^3 \geq \frac{1}{22}$.

Part (ii). Take $m_0 = 5$. We have $p = 1$ by definition. A general member $S \in |M_5|$ is a nonsingular projective surface of general type. Set $G := S|_S$.

If φ_5 is generically finite, then $\varphi_{|G|}$ is either birational or generically finite of degree ≥ 2. We have the following argument:

> (♯) If $\varphi_{|G|}$ gives a birational map, then clearly $h^0(S, G) \geq 4$ because S is of general type. Since $p_g(V) > 0$, we know $G \leq (K_{X'} + S)|_S = K_S$. And because G is nef, Lemma 2.1 of [6] says $C^2 \geq 3h^0(S, G) - 7 \geq 5$. When $|S|$ gives a generically finite map of degree ≥ 2, then Lemma 2.2 of [6] gives $C^2 \geq 2h^0(S, G) - 4$.

Because $h^0(S, G) \geq h^0(X', S) - 1 \geq 4$, the argument (♯) says $C^2 \geq 4$. We get $\deg(K_C) = (K_S + C)C > 2C^2 \geq 8$ noting that $K_{X'}|_S \cdot C \geq \pi^*(K_X)|_S \cdot C > 0$ by the Hodge Index Theorem. Actually we have $\deg(K_C) \geq 10$ since it is even. On the other hand, we can take $\beta = \frac{1}{5}$ since $5\pi^*(K_X)|_S \geq C$. Now take a very big m. Theorem 2.3 gives $\xi \geq \frac{10}{11}$. Take $m = 13$. Then $\alpha \geq \frac{20}{11} > 1$. We get $\xi \geq \frac{12}{13}$. So inequality (2.1) gives $\xi \geq \frac{12}{25 \cdot 13}$. In fact a similar calculation says $\xi \geq \frac{l}{l+1}$ for all $l \geq 12$. Thus $\xi \geq 1$ and (2.1) gives $K^3 \geq \frac{1}{25}$.

If $\dim(B) = 2$, we know that $|G| := |S|_S|$ is composed with a pencil of curves. A generic irreducible element C of $|G|$ is a smooth curve of genus ≥ 2, so $\deg(K_C) \geq 2$. Furthermore we have $h^0(S, G) \geq h^0(X', S) - 1 \geq 4$. So $G \equiv \widetilde{a}C$ for $\widetilde{a} \geq h^0(S, G) - 1 \geq 3$. This means $5\pi^*(K_X)|_S \geq S|_S \geq_{\text{numerically}} 3C$. So we may take $\beta = \frac{3}{5}$. Now take a very big m. Theorem 2.3 gives $\xi \geq \frac{6}{23}$. Take $m = 12$. Then $\alpha \geq \frac{26}{23} > 1$. We get $\xi \geq \frac{1}{3}$. Take $m = 11$. Then $\alpha \geq \frac{10}{9} > 1$. We get $\xi \geq \frac{4}{11}$. So inequality (2.1) gives $K^3 \geq \frac{1}{25} \cdot \frac{12}{11} > \frac{1}{25}$.

Part (iii). Take $m_0 = 5$. Parallel to the last parts in the proof of (i), we can discuss according to the value of $b = g(B)$. So we have more or less a redundant calculation as follows.

If $b = g(B) > 0$, there is an induced fibration $f : X' \longrightarrow B$. Because $p_g(V) > 0$, one sees $p_g(S) > 0$. One has $p = a_5 \geq P_5 \geq 5$ by the

Riemann-Roch and Clifford's theorem. Set $G := 2\sigma^*(K_{S_0})$. Because $|2K_{S_0}|$ is base point free, $|G| = |2\sigma^*(K_{S_0})|$ is also base point free. Denote by C a generic irreducible element of $|G|$. Then C is smooth and $\deg(K_C) = (K_S+C)C > 4\sigma^*(K_{S_0})^2 \geq 4$. So actually $\deg(K_C) \geq 6$. By Remark 2.4(2), we can take $\beta = \frac{1}{2}$ since $\pi^*(K_X)|_S \sim \frac{1}{2}(2\sigma^*(K_{S_0}))$. Now if we take a very big m, Theorem 2.3 gives $\xi \geq \frac{3}{2}$. Inequality (2.1) gives $K^3 \geq \frac{3}{4}$.

If $b = g(B) = 0$, we have $p \geq P_5 - 1 \geq 4$. Take $m_0 = 5$. Because $p_g(V) > 0$, one sees $p_g(S) > 0$. We still set $G := 2\sigma^*(K_{S_0})$. We have $\deg(K_C) \geq 6$. Similarly we only have to find a suitable β. Remark 2.4(3) says that one can find a sequence of positive rational numbers $\{\widetilde{\beta}_n\}$ with $\widetilde{\beta}_n \mapsto \frac{p}{p+5} \geq \frac{4}{9}$ such that $\pi^*(K_X)|_S - \widetilde{\beta}_n\sigma^*(K_{S_0})$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. Set $\beta_n := \frac{1}{2}\widetilde{\beta}_n$. Then $\pi^*(K_X)|_S - \beta_n C$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. We know $\beta_n \mapsto \frac{2}{9}$ whenever $p = 4$. When m is very big, Theorem 2.3 gives $\xi \geq \frac{8}{9}$. Take $m = 8$. Then $\alpha \geq \frac{10}{9} > 1$. We get $\xi \geq 1$. So inequality (2.1) gives $K^3 \geq \frac{8}{45}$. This completes the proof. □

Remark 2.6. One may remove extra condition: $p_g(V) > 0$ in Theorem 2.5 (ii) and (iii) to obtain parallel, but weaker results. We omit the details simply because it is not used in the proof of the main theorem.

3. On irregular 3-folds of general type

In this section, we study the canonical volume of irregular threefolds of general type. Let X be a nonsingular projective threefold of general type and $a : X \to \mathrm{alb}(X)$ the Albanese map. By running the minimal model program, one easily see that the Albanese map factors through its minimal model. So we may and do assume that X is a minimal (K_X nef) threefold of general type with $\mathbb{Q}$-factorial terminal singularities.

In the study of pluricanonical systems on irregular threefolds, the most unpleasant case is when the Albanese map $a : X \to \mathrm{Alb}(X)$ is surjective onto an elliptic curve E with general fiber F of type $(K_F^2, p_g(F)) = (1, 2)$.

Theorem 3.1. *Let X be a minimal 3-fold of general type with $q(X) = 1$ and the general fiber of $a : X \to \mathrm{Alb}(X)$ is of $(1, 2)$ type. Then the canonical volume $Vol(X) = K_X^3 \geq \frac{1}{9}$.*

Before proving the main result, we would like to recall some notion and results in [4].

Definition 3.2. For any vector bundle E on an elliptic curve, we write $E = \oplus E_i$ for its decomposition into indecomposable vector bundles. We define $\nu(E) := \min \mu(E_i)$, where $\mu(E_i) = \frac{deg(E_i)}{rk(E_i)}$.

Lemma 3.3. *([4], Lemma 4.8) Let E_1, E_2 be indecomposable vector bundles on an elliptic curve. If $\mathrm{Hom}(E_1, E_2) \neq 0$, then $\mu(E_2) \geq \mu(E_1)$.*

In particular $\nu(E_2) \geq \nu(E_1)$ if $E_1 \to E_2$ is a surjective map of vector bundles.

Definition 3.4. A coherent sheaf $\mathcal{F}$ on an abelian variety A is said to be IT^0 if $H^i(A, \mathcal{F} \otimes P) = 0$ for all $i > 0$ and all $P \in \mathrm{Pic}^0(A)$.

Lemma 3.5. *([4], Lemma 4.10) Let E be an IT^0 vector bundle on an elliptic curve which admits a short exact sequence*

$$0 \to F \to E \to Q \to 0$$

of coherent sheaves such that Q has generic rank ≤ 1. Then $\nu(E) \geq \min\{1, \nu(F)\}$.

3.6. Multiplication maps $\varphi_{m,n}$ and $\psi_{m,n}$. Let $R_m := H^0(F, \omega_F^m)$ and $E_m := a_*\omega_X^m$. By Lemma 4.1 of [4], E_m is an IT^0 vector bundle of rank $P_m(F)$ for all $m \geq 2$. We also remark that $\nu(E_1) \geq 0$ by the semi-positivity theorem (see Viehweg [24]) and Atiyah's description of vector bundles over elliptic curves (cf. [1]). We consider the multiplication map of pluricanonical systems on fibers

$$\varphi_{m,n} : R_m \otimes R_n \to R_{m+n}.$$

This induces a map

$$\psi_{m,n} : E_m \otimes E_n \to E_{m+n}.$$

Clearly if cokernel of $\varphi_{m,n}$ has dimension $\leq r$, then cokernel of $\psi_{m,n}$ has rank $\leq r$.

3.7. Surfaces of (1,2) type. Let F be a nonsingular minimal projective surface of general type with $(K_F^2, p_g(F)) = (1, 2)$. It's well-known that $|K_F|$ has only one base point z and $|2K_F|$ is base point free (cf. [2]).

We recall the following result in §5 of [4].

Lemma 3.8. *([4], p353, line -6) Assume that a general fiber F of the fibration $a : X \to \mathrm{Alb}(X)$ is a surface of (1,2) type. Then $\varphi_{1,m-1}$: $R_1 \otimes R_{m-1} \mapsto R_1 R_{m-1} \subset R_m$ has codimension ≤ 1 for all $m \geq 1$ and $\varphi_{1,2}$ is surjective.*

(†) Clearly $R_1R_2 = R_3$ implies that R_2R_2, which contains $R_1R_1R_2 = R_1R_3$, has codimension ≤ 1 in R_4.

Moreover, we have the following:

Lemma 3.9. *Assume that a general fiber F of the fibration $a : X \to \mathrm{Alb}(X)$ is a surface of (1,2) type. Then the multiplication map $\varphi_{2,m-2}$: $R_2 \otimes R_{m-2} \mapsto R_m$ is surjective whenever $m \geq 8$ and has codimension ≤ 1 if $m = 7$.*

Proof. We follow Bombieri's argument on projective normality.

Fix two sections $s, s_1 \in R_2$ with smooth curves $C := \mathrm{div}(s)$, $C_1 := \mathrm{div}(s_1)$ and assume that $Z := div(s_1) \cap C$ consists of 4 distinct points, we have the exact sequences:

$$0 \to R_{m-2} \xrightarrow{s} R_m \xrightarrow{r_C} H^0(C, mK_F|_C) \to 0, \tag{3.1}$$

$$0 \to R_{m-2} \xrightarrow{s_1} R_m \xrightarrow{r_{C_1}} H^0(C_1, mK_F|_{C_1}) \to 0, \tag{3.2}$$

$$0 \to R_{m-4} \xrightarrow{s} R_{m-2} \xrightarrow{r_C} H^0(C, (m-2)K_F|_C) \to 0, \tag{3.3}$$

thanks to the vanishing of $H^1(F, tK_F)$ for all $t \geq 0$. We also have the following exact sequence:

$$0 \to H^0(C, (m-2)K_F|_C) \xrightarrow{\widetilde{s}_1} H^0(C, mK_F|_C) \xrightarrow{r_Z} H^0(Z, \mathcal{O}_Z). \tag{3.4}$$

Among the four exact sequences, one can find the commutative relation: $\widetilde{s}_1 \circ r_C = r_C \circ s_1$.

$$\begin{array}{ccc} R_{m-2} & \xrightarrow{s_1} & R_m \\ {\scriptstyle r_C}\downarrow & & \downarrow{\scriptstyle r_C} \\ H^0(C, (m-2)K_F|_C) & \xrightarrow[\widetilde{s}_1]{} & H^0(C, mK_F|_C) \end{array}$$

One knows $\widetilde{s}_1 \circ r_C(R_{m-2}) = \widetilde{s}_1 H^0(C, (m-2)K_F|_C)$ has codimension ≤ 4 in the space $H^0(C, mK_F|_C)$ since $\dim H^0(Z, \mathcal{O}_Z) = 4$. So $r_C \circ s_1(R_{m-2})$ has codimension $d' \leq 4$ in $H^0(C, mK_F|_C)$. Since

$$\begin{aligned} d' &= h^0(C, mK_F|_C) - \dim r_C(s_1 R_{m-2}) \\ &\geq \dim R_m - \dim sR_{m-2} - \dim s_1 R_{m-2}. \end{aligned}$$

It follows that $sR_{m-2} + s_1 R_{m-2} \subset R_m$ has codimension ≤ 4.

Moreover, we consider

$$\begin{aligned} 0 &\to H^0(C, (m-4)K_F|_C) \xrightarrow{\widetilde{s}_1} H^0(C, (m-2)K_F|_C) \\ &\xrightarrow{r_Z} H^0(Z, \mathcal{O}_Z) \to H^1(C, (m-4)K_F|_C). \end{aligned}$$

Since $3K_{F|C} = K_C$ and $\deg(K_F|_C) = 2$, one sees that $H^1(C, (m-4)K_F|_C) = 0$ if $m \geq 8$. When $m = 7$, $H^1(C, 3K_F|_C) = H^1(C, K_C)$ is one dimensional. We can take a section $s_2 \in R_2$ such that s_2 never vanishing on Z. Set $J = div(s_2) \cap C$ which can be a union of 4 distinct points. As we have seen the map $r_J : r_C(s_2 R_{m-2}) \to H^0(J, \mathcal{O}_J)$ is either surjective when $m \geq 8$ or having codimension ≤ 1 when $m = 7$. Together with surjectivity of r_C, we see that $r_C(s_2 R_{m-2}) = \widetilde{s}_2(H^0(C, (m-2)K_F|_C))$ where

$$\widetilde{s}_2 : H^0(C, (m-2)K_F|_C) \mapsto H^0(C, mK_F|_C)$$

is defined by the multiplication of s_2. This already means that

$$sR_{m-2} + s_1 R_{m-2} + s_2 R_{m-2} \subset R_m$$

has codimension 0 or ≤ 1 if $m \geq 8$ or $= 7$ respectively. We are done. $\square$

Now we prove Theorem 3.1.

Proof of Theorem 3.1. First of all, E_2 is an IT^0 vector bundle of rank 4. So one has $\nu(E_2) \geq \frac{1}{4}$.

Consider the induced multiplication map $\psi_{2,2} : E_2 \otimes E_2 \to E_4$. Since $\varphi_{2,2}$ has image of codimension ≤ 1 by (†), it follows that $\psi_{2,2}$ has cokernel of rank ≤ 1. We consider the exact sequence

$$0 \to \mathrm{Im}(\psi_{2,2}) \to E_4 \to \mathrm{Coker}(\psi_{2,2}) \to 0.$$

By Lemma 3.5 and Lemma 3.3, one has

$$\begin{aligned} \mu(E_4) &\geq \min\{\nu(\mathrm{Im}(\psi_{2,2})), 1\} \geq \min\{\nu(E_2 \otimes E_2), 1\} \\ &= \min\{2\nu(E_2), 1\} \geq \frac{1}{2}. \end{aligned}$$

Next, similarly, we consider $\psi_{4,1}$, then we see that

$$\nu(E_5) \geq \min\{\nu(E_4) + \nu(E_1), 1\} \geq \frac{1}{2}.$$

By considering $\psi_{5,2}$, we see that $\nu(E_7) \geq \min\{\nu(E_5) + \nu(E_2), 1\} \geq \frac{3}{4}$.

Finally, we consider $\psi_{7,2}$, then we have $\nu(E_9) \geq 1$.

Now $\nu(E_9) \geq 1$ implies that there is a line bundle L of degree 1 with an injection $L \to E_9$. In particular, $H^0(X, 9K_X \otimes f^* L^\vee)$ has a section. Thus $9K_X \geq F$. Hence

$$9K_X^3 \geq K_X^2 \cdot F = K_F^2 = 1.$$

This completes the proof. □

Next we consider the case with $(K_F^2, p_g(F)) = (2, 3)$. In this case, $\varphi_{2,m-2} : R_2 \otimes R_{m-2} \to R_m$ is surjective for $m \geq 6$ by the same argument(cf. remark at p190, [2]). Also one can check that $\varphi_{2,2} : R_2 \otimes R_2 \to R_4$ has cokernel of dimension ≤ 1. Thus we are able to show the following:

Proposition 3.10. *Let X be a minimal 3-fold of general type with $q(X) = 1$ and the general fiber of $a : X \to \mathrm{Alb}(X)$ is of $(2, 3)$ type. Then the canonical volume Vol$(X) \geq \frac{1}{6}$.*

Proof. We have $\nu(E_2) \geq \frac{1}{6}$. By considering $\psi_{2,2}$, we have $\nu(E_4) \geq \min\{2\nu(E_2), 1\} \geq \frac{1}{3}$. Then consider $\psi_{2,4}, ..., \psi_{2,10}$ inductively, we get $\nu(E_{12}) \geq 1$. Hence $12K_X > F$. So we have

$$12K_X^3 \geq K_X^2 \cdot F = K_F^2 = 2.$$

□

Combining Theorem 3.1, results in [4], and Theorem 2.5, we are able to get a lower bound of the canonical volume for all those irregular threefolds.

Corollary 3.11. *Let V be a nonsingular projective irregular 3-fold of general type. Then* $\mathrm{Vol}(V) \geq \frac{1}{22}$.

Proof. We consider 3-folds of general type with $q(V) > 0$. Then there is a non-trivial Albanese map $a : V \to \mathrm{Alb}(V)$. If the general fiber has dimension ≤ 1, then by Proposition 2.9 of [4], $|4K_V + P|$ is birational for general $P \in \mathrm{Pic}^0(V)$. In particular, $h^0(V, \mathcal{O}_V(4K_V)\otimes P) \geq 4$. However, it's in fact ≥ 5 because otherwise it gives a birational map onto $\mathbb{P}^3$, which is not of general type. By the upper semicontinuity of cohomology, we have $P_4(V) = h^0(V, \mathcal{O}(4K_V)) \geq 5$. Now by Theorem 2.5 (i), we get $\mathrm{Vol}(V) \geq \frac{1}{22}$.

We now assume that the Albanese map has 1-dimensional image. Let $f : V \to H$ be an induced fibration from the Stein factorization of a. We now consider the case that $g(H) \geq 2$. We can take the relative minimal model of f, say $h : X \to H$. So X is birational to V. By Theorem 1.4 of [17], $K_{X/H} := K_X - h^*(K_H)$ is nef. In particular K_X is nef and X is minimal. Because $g(H) \geq 2$, we see that $K_X - 2F$ is nef where F is a general fiber of h. So $\mathrm{Vol}(V) = K_X^3 \geq 2K_F^2 \geq 2 > \frac{1}{22}$.

Finally, we consider the case that $g(H) = 1$. We remark that $g(H) = 1$ if and only if $q(V) = 1$ because if $q(V) \geq 2$, then either its Albanese image has dimension ≥ 2 or is a curve of genus ≥ 2.

If F is not of the type $(1, 2)$, then $|4K_F|$ is birational according to Bombieri's classification. By Theorem 2.8 of [4], $|4K_V+P|$ is birational for general $P \in \mathrm{Pic}^0(X)$. So we get $\mathrm{Vol}(V) \geq \frac{1}{22}$ as above.

It remains to consider the case that F is of type (1.2). By Theorem 3.1, we have $\mathrm{Vol}(V) \geq \frac{1}{9} > \frac{1}{22}$. □

4. The case $P_5 < 5$

4.1. First let us recall Reid's plurigenera formula (cf. [18], p413) for a minimal 3-fold X of general type (with $\mathbb{Q}$-factorial terminal singularities):

$$P_m(X) = \frac{1}{12}m(m-1)(2m-1)K_X^3 - (2m-1)\chi(\mathcal{O}_X) + l(m) \quad (4.1)$$

where m is an integer > 1. The correction term is

$$l(m) := \sum_Q l_Q(m) := \sum_Q \sum_{j=1}^{m-1} \frac{\overline{bj}(r - \overline{bj})}{2r},$$

where the sum $\sum_Q$ runs through all baskets Q of singularities of type $\frac{1}{r}(a, -a, 1)$ with the integer a coprime to r, $0 < a < r$, $0 < b < r$, $ab \equiv 1 \pmod r$, $\overline{bj}$ the smallest residue of bj mod r. One can see easily that $(b, r) = 1$. Note by definition that the singularity $\frac{1}{r}(a, -a, 1)$ is a terminal quotient one obtained by a cyclic group action on $(\mathbb{C}^3, (0, 0, 0))$:

$$\varepsilon(x, y, z) = (\varepsilon^a x, \varepsilon^{-a} y, z)$$

where ε is a fixed r-th primitive root of 1. Reid's Theorem 10.2 in [18] says that the above baskets {Q} of singularities are in fact virtual (!) and that one need not worry about the authentic type of all those terminal singularities on X, though X may have non-quotient terminal singularities. Iano-Fletcher [13] has shown that the set of baskets {Q} in Reid's formula is uniquely determined by X.

In the next context we will always study those 3-folds X with the following conditions:

(*) $p_g = 1$, $\chi(\mathcal{O}) = 0$ and $P_5 \leq 4$.

4.2. Reid's formula (4.1) tells:

$$P_5 > P_4 > P_3 > P_2 > 0$$

whenever $\chi(\mathcal{O}) = 0$. So one gets $P_2 = 1$, $P_3 = 2$, $P_4 = 3$ and $P_5 = 4$.

We shall classify those X satisfying the condition that $\chi(\mathcal{O}_X) = 0, P_2(X) = 1, P_3(X) = 2$.

Recall the plurigenera formula that

$$P_m(X) = \frac{1}{12}m(m-1)(2m-1)K_X^3 + \sum_Q \sum_{j=1}^{m-1} \frac{\overline{b_Q j}(r_Q - \overline{b_Q j})}{2r_Q}.$$

We introduce

$$b'_Q := \begin{cases} b_Q, & \text{if} \quad b_Q \leq \frac{1}{2}r_Q; \\ r_Q - b_Q, & \text{if} \quad b_Q > \frac{1}{2}r_Q. \end{cases}$$

Then it's easy to see that $\overline{b_Q j}(r_Q - \overline{b_Q j}) = b'_Q j(r_Q - b'_Q j)$ for $j = 1, 2$.

For $m = 2, 3$, we have

$$1 = P_2(X) = \frac{1}{2}K_X^3 + \sum_Q \frac{b'_Q(r_Q - b'_Q)}{2r_Q} = \frac{1}{2}K_X^3 + \frac{1}{2}\sum_Q b'_Q - \frac{1}{2}\sum_Q \frac{b'^2_Q}{r_Q},$$

$$2 = P_3(X) = \frac{5}{2}K_X^3 + \frac{3}{2}\sum_Q b'_Q - \frac{5}{2}\sum_Q \frac{b'^2_Q}{r_Q}.$$

By solving these, we get

$$\sum_Q b'_Q = 3, \quad \sum_Q \frac{b'^2_Q}{r_Q} = 1 + K_X^3. \tag{4.1}$$

Moreover, the inequality

$$1 = P_2(X) \geq \frac{1}{2}K^3 + \sum \frac{r-1}{2r} > \frac{n}{4}$$

implies $n < 4$, where n denotes the number of baskets. Thus $n = 3, 2, 1$.

4.3. Three basket case. First we consider the case $n = 3$. Assume that the basket Q_i is of the type $\frac{1}{r_i}(a_i, -a_i, 1)$ with $a_i b_i \equiv 1 \pmod{r_i}$ and $0 < b_i < r_i$ for $i = 1, 2, 3$. Since $K^3 > 0$, one has:

$$1 = P_2(X) > \sum_{i=1}^{3} \frac{b_i(r_i - b_i)}{2r_i} \geq \sum_{i=1}^{3} \frac{r_i - 1}{2r_i}$$

and so

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}>1. \tag{4.2}$$

One may assume $r_1 \le r_2 \le r_3$. Then clearly, the only possible solution for (r_1, r_2, r_3) are $(2,3,3), (2,3,4), (2,3,5)$ and $(2,2,r_3)$.

(4.3.1.) The case $(r_1, r_2, r_3) = (2,2,r_3)$.
By (4.1), we have $b_1' = b_2' = b_3' = 1$ and $K_X^3 = \frac{1}{r_3}$. Hence $b_1 = b_2 = 1$, and $b_3 = 1$ or $r_3 - 1$. Easy computation shows that $P_4(X) = 4$ (resp. $= 5$) if $r_3 \ge 3$ (resp. $r_3 = 2$). And also $P_5(X) = 6$ (resp. $= 7, = 9$) if $r_3 \ge 4$ (resp. $r_3 = 3, = 2$).

(4.3.2.) The case $(r_1, r_2, r_3) = (2,3,3)$.
Computation shows that $K_X^3 = \frac{1}{6}$ and $P_4 = 3, P_5 = 5$.

(4.3.3.) The case $(r_1, r_2, r_3) = (2,3,4)$.
Since $b_1' = b_2' = b_3' = 1$. Then one gets a possible case:

> **(C1).** $(r_1, r_2, r_3) = (2,3,4)$, $K^3 = \frac{1}{12}$, $P_2 = 1$, $P_3 = 2$, $P_4 = 3$ and $P_5 = 4$.

(4.3.4.) The case $(r_1, r_2, r_3) = (2,3,5)$.
Similarly, by $b_1' = b_2' = b_3' = 1$. So we have found another possible case:

> **(C2).** $(r_1, r_2, r_3) = (2,3,5)$, $b_3 = 1$ or 4, $K^3 = \frac{1}{30}$, $P_2 = 1$, $P_3 = 2$, $P_4 = 3$ and $P_5 = 4$.

4.4. Two basket case. Consider the case $n = 2$. We may assume $r_1 \le r_2$. Also recall that $b_1' + b_2' = 3$ by (4.1). We will distinquish the following two cases.

(4.4.1.) $b_1' = 1, b_2' = 2$.
By (4.1), $1 + K_X^3 = \frac{1}{r_1} + \frac{4}{r_2}$. Hence we have $\frac{5}{r_1} \ge \frac{1}{r_1} + \frac{4}{r_2} > 1$. It follows that $r_1 < 5$.

If $r_1 = 2$, then one gets $\frac{4}{r_2} = K^3 + \frac{1}{2}$ and hence $r_2 < 8$. Noting $(b_2, r_2) = 1$ and $b_2' \le \frac{1}{2} r_2$, one sees that $r_2 = 5, 7$. Whenever $(r_1, r_2) = (2,5)$, then computation shows that $K_X^3 = \frac{3}{10}, P_4 = 4, P_5 = 7$. Whenever $(r_1, r_2) = (2,7)$, we have found the possible case:

> **(C3).** $(r_1, r_2) = (2,7)$, $b_2 = 2$ or 5, $K^3 = \frac{1}{14}$, $P_2 = 1$, $P_3 = 2$, $P_4 = 3$ and $P_5 = 4$.

If $r_1 = 3$, then $\frac{4}{r_2} = \frac{2}{3} + K_X^3 > \frac{2}{3}$. This gives $r_2 < 6$. The only possibility is $(r_1, r_2) = (3,5)$ since $2 = b_2' \le \frac{1}{2} r_2$ and $(b_2', r_2) = 1$. Computation shows that $K_X^3 = \frac{2}{15}, P_4 = 3$ and $P_5 = 5$.

If $r_1 = 4$, then similarly we have $r_2 \le 5$. The only possibility is $r_2 = 5$. So $K^3 = \frac{1}{20}$ and $P_3 = 2$, $P_4 = 3$, $P_5 = 4$. We have found the possible case:

> **(C4).** $(r_1, r_2) = (4,5)$, $b_2 = 2$ or 3, $K^3 = \frac{1}{20}$, $P_2 = 1$, $P_3 = 2$, $P_4 = 3$ and $P_5 = 4$.

(4.4.2.) $b'_1 = 2, b'_2 = 1$.
By (4.1), $1 + K_X^3 = \frac{4}{r_1} + \frac{1}{r_2}$. Hence we have $\frac{5}{r_1} \geq \frac{1}{r_1} + \frac{4}{r_2} > 1$. It follows that $r_1 < 5$. However, $2 = b'_1 \leq \frac{1}{2} r_1$ and $(2, r_1) = 1$ gives a contradiction.

4.5. One basket case. By (4.1), one has $b' = 3$ and $\frac{9}{r} = 1 + K_X^3 > 1$. Hence $r < 9$. Moreover $b' \leq \frac{r}{2}$ and $(b', r) = 1$, so it follows that $r = 7, 8$.
If $r = 7$, one gets $K_X^3 = \frac{2}{7}, P_4 = 4, P_5 = 7$.
If $r = 8$, one gets $K_X^3 = \frac{1}{8}, P_4 = 3, P_5 = 5$.

We summarize all the possible cases with $P_g = 1, P_5 < 5$:

Corollary 4.6. *Let X be a minimal projective 3-fold of general type with $\chi(\mathcal{O}_X) = 0$, $P_g(X) = 1$ and $P_5(X) = 4$. Then X has at most 3 baskets of singularities of type $\frac{1}{r}(a, -a, 1)$ and one of the following 4 situations occurs:*
(C1). $(r_1, r_2, r_3) = (2, 3, 4)$, $K^3 = \frac{1}{12}$;
(C2). $(r_1, r_2, r_3) = (2, 3, 5)$, $b_3 = 1$ *or* 4, $K^3 = \frac{1}{30}$;
(C3). $(r_1, r_2) = (2, 7)$, $b_2 = 2$ *or* 5, $K^3 = \frac{1}{14}$;
(C4). $(r_1, r_2) = (4, 5)$, $b_2 = 2$ *or* 3, $K^3 = \frac{1}{20}$.

Example 1.2 shows that the situation **(C2)** really do occur. We give another example to show the existence of **(C3), (C4)**.

Example 4.7. (1) ([10], p153) The canonical hypersurface
$$X_{12,15} \subset \mathbb{P}(1, 3, 4, 5, 6, 7)$$
has two terminal singularities: $1 \times \frac{1}{7}(4, -4, 1)$, $1 \times \frac{1}{2}(1, -1, 1)$. The canonical volume is $\frac{1}{14}$. This example corresponds to **(C3)**.
(2) ([10], p151) The canonical hypersurface
$$X_{21} \subset \mathbb{P}(1, 3, 4, 5, 7)$$
has two terminal singularities: $1 \times \frac{1}{4}(1, -1, 1)$, $1 \times \frac{1}{5}(3, -3, 1)$. The canonical volume is $\frac{1}{20}$. This example corresponds to **(C4)**.

It is interesting to ask:

Question 4.8. Does **(C1)** really occur?

4.9. Proof of Theorem 1.1.

Proof. Let X be a minimal projective 3-fold of general type (admitting at worst $\mathbb{Q}$-factorial terminal singularities) with $\chi(\mathcal{O}_X) \leq 0$. Recall that one has
$$\chi(\mathcal{O}_X) = 1 - q + h^2(\mathcal{O}_X) - p_g$$
where the irregularity $q := h^1(\mathcal{O}_X)$ and the geometric genus $p_g := h^3(\mathcal{O}_X)$. Since $\mathrm{Vol}(X) \geq \frac{1}{3}$ whenever $p_g \geq 2$ by [5], and $\mathrm{Vol}(X) \geq \frac{1}{22}$ whenever $q > 0$ by Corollary 3.11, we may assume, from now on, that $p_g \leq 1$ and $q = 0$. Therefore the assumption $\chi(\mathcal{O}_X) \leq 0$ implies $p_g = 1$, $h^2(\mathcal{O}_X) = 0$ and finally $\chi(\mathcal{O}_X) = 0$.

Whenever $P_5(X) \geq 5$, Theorem 2.5 (ii) and (iii) says $\mathrm{Vol}(X) > \frac{1}{25}$.

Whenever $P_5(X) \leq 4$, $p_g > 0$ and $\chi(\mathcal{O}_X) = 0$, Corollary 4.6 says that $\mathrm{Vol}(X) \geq \frac{1}{30}$. Furthermore $\mathrm{Vol}(X) = \frac{1}{30}$ implies that X corresponds exactly to the situation (**C2**) in the list of Corollary 4.6. This completes the proof. □

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