Advanced Algebra II Homework 4

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1. * Complete the exercises and incomplete proofs in the note.

2. Let M be a Noetherian R-module, and let $\mathfrak{a} \lhd M$ be the annihilator of M. Prove that M is a Noetherian $R/\mathfrak{a}$-module. How about if we replace Noetherian by Artinian?

   Proof.
   Consider N be any $R/\mathfrak{a}-$submodule of M, then for any $r \in R,\ n \in N$, $rn = (r + \mathfrak{a})\, n \in N$, so N is a $R-$submodule of M too. Therefore, the ACC or DCC on set of all $R/\mathfrak{a}-$submodules of M follows from the ACC or DCC on set of all $R-$submodules of M, hence M is a Noetherian (Artinian) $R/\mathfrak{a}$-module if M is Noetherian (Artinian) R-module. ■

3. * Let R be a Noetherian local ring and M be a finitely generated R-module. Show that M is free if and only if M is flat.

   Proof.
   ($\Leftarrow$) Apply my proof of problem 3 in Homework 3, then we are done.
   ($\Rightarrow$) Suppose M is free, then M has basis $\{x_1, ..., x_n\}$, so given any $\mathfrak{a} \lhd R$, each element of $\mathfrak{a} \otimes M$ has the form $\sum_{i=1}^{n} a_i \otimes x_i$ for $a_i \in \mathfrak{a}$. Let $\varphi : \mathfrak{a} \otimes M \to \mathfrak{a}M$ given by $a \otimes x \to ax$, then we have
   $$\varphi\left(\sum_{i=1}^{n} a_i \otimes x_i\right) = \sum_{i=1}^{n} a_i x_i = 0 \Rightarrow a_i = 0\ \forall i$$
   because $\{x_i\}$ is basis of M, so φ is injective and of course surjective. It's clear that the sequence
   $$0 \to \mathfrak{a}M \cong \mathfrak{a} \otimes M \to M \cong R \otimes M$$
   is exact, so M is flat. ■

4. Let R be an Noetherian ring, and $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal. Show that there exists $n \geq 1$ such that $\mathfrak{p}^n \subseteq \mathfrak{q}$.
   Is it still true if R is not necessarily Noetherian?

   Proof.
   Since R is Noetherian, $\mathfrak{p}$ is finitely generated, say by $\{x_1, ..., x_k\}$, then
   $$x_i \in \mathfrak{p} = \sqrt{\mathfrak{q}} \Rightarrow \exists m_i \in \mathbb{N} \ni x_i^{m_i} \in \mathfrak{q}.$$
   Let $n = m_1 + \cdots + m_k$, given any $x = \sum_i \prod_{j=1}^{n} p_{ij} \in \mathfrak{p}^n$, write $p_{ij} = \sum_{l=1}^{k} r_{ijl} x_l$, then
   $$x = \sum_i \prod_{j=1}^{n} \sum_{l=1}^{k} r_{ijl} x_l = \sum_i \left(\sum_{\varphi_1 + \cdots + \varphi_k = n} r'_{\varphi_1 \cdots \varphi_k} x_1^{\varphi_1} \cdots x_n^{\varphi_k} \right).$$
   If $\varphi_l < m_l$ for all l, then $\sum_l \varphi_l = n < \sum_l m_l = n$, a contradiction. Therefore $\exists l \ni \varphi_l \geq m_l$, and
   $$x_l^{\varphi_l} = x_l^{\varphi_l - m_l} x_l^{m_l} \in \mathfrak{q},$$
   hence all the summand $r'_{\varphi_1 \cdots \varphi_k} x_1^{\varphi_1} \cdots x_n^{\varphi_k} \in \mathfrak{q}$, $x \in \mathfrak{q}$, $\mathfrak{p}^n \subseteq \mathfrak{q}$.
   If R is not Noetherian, then the statement may not hold. Consider $R = K[x_1, x_2, x_3, ...]$ be the polynomial ring with infinite indeterminates over a field K, then clearly $\mathfrak{p} \equiv (x_1, x_3, ..., x_{2n+1}, ...)$ is prime and $\mathfrak{q} \equiv \left(x_1, x_3^3, ..., x_{2n+1}^{2n+1}, ...\right)$ is $\mathfrak{p}-$primary, but $\left(x_{2n+1}^2\right)^n \in \mathfrak{p}^n - \mathfrak{q}$ for all n, so $\mathfrak{p}^n \nsubseteq \mathfrak{q}$ for all n. ■

5. Let $f : A \to B$ be a homomorphism between local rings $(A, \mathfrak{m}_A), (B, \mathfrak{m}_B)$. We say that f is **local** if $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$. If we start with a homomorphism $f : A \to B$ of rings. For any $\mathfrak{q} \in \mathrm{Spec}(B)$, we have $\mathfrak{p} = f^{-1}(\mathfrak{q}) \in \mathrm{Spec}(A)$. Show that the induced map $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is local.

Proof.

Let $\mathfrak{p} = f^{-1}(\mathfrak{q})$, then given $ab \in \mathfrak{p}$,

$$f(ab) = f(a) f(b) \in \mathfrak{q} \Rightarrow f(a) \in \mathfrak{q} \ \text{ or } \ f(b) \in \mathfrak{q} \Rightarrow a \in f^{-1}(\mathfrak{q}) = \mathfrak{p} \ \text{ or } \ b \in f^{-1}(\mathfrak{q}) = \mathfrak{p},$$

so $\mathfrak{p} \in \mathrm{Spec}(A)$.

The induced map $\bar{f} : A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is defined by

$$\bar{f}\left(\frac{x}{s}\right) = \frac{f(x)}{f(s)} \qquad \forall \frac{x}{s} \in A_{\mathfrak{p}}.$$

It's easy to verify that $\bar{f}$ is well-defined because $s \notin \mathfrak{p} = f^{-1}(\mathfrak{q}) \Rightarrow f(s) \notin \mathfrak{q}$. Now we shall prove that $\bar{f}^{-1}(\mathfrak{m}_{B_{\mathfrak{q}}}) = \mathfrak{m}_{A_{\mathfrak{p}}}$.

Given $\dfrac{x}{s} \in \bar{f}^{-1}(\mathfrak{m}_{B_{\mathfrak{q}}})$ for $x \in A$, $s \in \mathfrak{p}^c$, we have

$$\bar{f}\left(\frac{x}{s}\right) = \frac{f(x)}{f(s)} \in \mathfrak{m}_{B_{\mathfrak{q}}} = (\mathfrak{q}^c)^{-1}\mathfrak{q} \Rightarrow \exists y \in \mathfrak{q},\ s', s'' \in \mathfrak{q}^c \ \ni s''\big(f(x)s' - f(s)y\big) = 0,$$

that is,

$$s's''f(x) = s''f(s)y \in \mathfrak{q} \Rightarrow f(x) \in \mathfrak{q} \quad \because s', s'' \in \mathfrak{q}^c,\ \mathfrak{q} \in \mathrm{Spec}(B).$$

Hence $x \in f^{-1}(\mathfrak{q}) = \mathfrak{p}$, so $\dfrac{x}{s} \in (\mathfrak{p}^c)^{-1}\mathfrak{p} = \mathfrak{m}_{A_{\mathfrak{p}}}$.

Conversely, given $\dfrac{x}{s} \in \mathfrak{m}_{A_{\mathfrak{p}}}$ for $x \in \mathfrak{p} = f^{-1}(\mathfrak{q})$, $s \in \mathfrak{p}^c$, then $f(x) \in \mathfrak{q}$ and $f(s) \in \mathfrak{q}^c$, so clearly $\bar{f}\left(\dfrac{x}{s}\right) = \dfrac{f(x)}{f(s)} \in (\mathfrak{q}^c)^{-1}\mathfrak{q} = \mathfrak{m}_{B_{\mathfrak{q}}}$. ■

6. Let K be an algebraically closed field. Consider the ring homomorphism $f : A \equiv K[x] \to B \equiv K[x,y]/(y^2 - x)$ which sends $f(x) = x$.

   (a) Show that B is integral over A.

   (b) For each prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$, determine the prime ideals of B lying over $\mathfrak{p}$.

   (c) Show that for each prime ideal $\mathfrak{q} \in \mathrm{Spec}(B)$, lying over $\mathfrak{p}$, we have a local homomorphism $(A_{\mathfrak{p}}, \mathfrak{m}_{\mathfrak{p}}) \to (B_{\mathfrak{q}}, \mathfrak{m}_{\mathfrak{q}})$. Moreover, a K-vector space homomorphism $f_{\mathfrak{q}} : \mathfrak{m}_{\mathfrak{p}}/(\mathfrak{m}_{\mathfrak{p}})^2 \to \mathfrak{m}_{\mathfrak{q}}/(\mathfrak{m}_{\mathfrak{q}})^2$.

   (d) Show that for $\mathfrak{q} \neq 0$, all the above vector space $\mathfrak{m}_{\mathfrak{p}}/(\mathfrak{m}_{\mathfrak{p}})^2, \mathfrak{m}_{\mathfrak{q}}/(\mathfrak{m}_{\mathfrak{q}})^2$ has dimension 1. And also determine when $f_{\mathfrak{q}}$ is not isomorphism.

Proof.

f actually maps x to $\bar{x} = x + (y^2 - x)$. Since f is injective, A can be embedded in B and moreover, we can admit the convention: denote x, y instead of $\bar{x}$, $\bar{y}$ in B, then B becomes the ring $K[x,y]$ providing $x = y^2$.

(a) y in B satisfies the monic polynomial $z^2 - x \in A[z]$ by definition, so y is integral over A, $A[y]$ is a finitely generated $A-$module by Proposition 1.7.1 (2). $u \in B = A[y] \Rightarrow A[u] \subseteq A[y]$, so Proposition 1.7.1(3) implies u is integral over A for all $u \in B$. B is integral over A.

(b) Since $A = K[x]$ is a PID, nonzero prime elements coincide with irreducible elements; since K is algebraic closed, irreducible polynomials are those linear factors, hence

$$\mathrm{Spec}(A) = \{A(x - u) \,|\, u \in K\} \cup \{0\}.$$

Note that $B = K[x, y] = K[y^2, y] = K[y]$, so

$$\operatorname{Spec}(B) = \{B(y-v) \mid v \in K\} \cup \{0\}.$$

If $y - v | g(x)$, then

$$y + v = -(-y-v) | g\left((-y)^2\right) = g\left(y^2\right) = g(x),$$

so $y^2 - v^2 | g(x)$. Therefore, given $\mathfrak{p} = A(x-u) \in \operatorname{Spec}(A)$,

$$B\left(y \pm \sqrt{u}\right) \cap A = A\left(y^2 - \left(\sqrt{u}\right)^2\right) = A(x-u),$$

so $B(y \pm \sqrt{u})$ are lying over $\mathfrak{p}$. If $\mathfrak{p} = 0$, then the only choice is $\mathfrak{q} = 0$.

(c) Consider 1_A in problem 5, then $1_A^{-1}(\mathfrak{q}) = \mathfrak{q} \cap A = \mathfrak{p}$, so we have a local homomorphism $(A_\mathfrak{p}, \mathfrak{m}_\mathfrak{p}) \to (B_\mathfrak{q}, \mathfrak{m}_\mathfrak{q})$ defined by natural inclusion φ.
$\mathfrak{m}_\mathfrak{p}/(\mathfrak{m}_\mathfrak{p})^2$ and $\mathfrak{m}_\mathfrak{q}/(\mathfrak{m}_\mathfrak{q})^2$ are $A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}-$module and $B_\mathfrak{q}/\mathfrak{m}_\mathfrak{q}-$module respectively. Let $QF(R)$ denote the quotient field of some ring R, and let $\mathfrak{q} = B(y-v)$, $\mathfrak{p} = A\left(x - v^2\right)$. By problem 6 of Homework 2,

$$A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} \cong QF(A/\mathfrak{p}) = QF(K[x]/K[x](x-u)) \cong QF(K[u]) \cong QF(K) = K,$$

where the map is given by

$$\frac{g(x)}{h(x)} + \mathfrak{m}_\mathfrak{p} \to \frac{g\left(v^2\right)}{h\left(v^2\right)} \quad \text{for } g(x) \in A,\ h(x) \in A - \mathfrak{p}.$$

Similarly, $B_\mathfrak{q}/\mathfrak{m}_\mathfrak{q} \cong K$ with the map given by

$$\frac{\varphi(y)}{\psi(y)} + \mathfrak{m}_\mathfrak{q} \to \frac{\varphi(v)}{\psi(v)} \quad \text{for } \mathfrak{q} = B(x-v),\ \varphi(y) \in B,\ \psi(y) \in B - \mathfrak{q}.$$

Hence both $\mathfrak{m}_\mathfrak{p}/(\mathfrak{m}_\mathfrak{p})^2$ and $\mathfrak{m}_\mathfrak{q}/(\mathfrak{m}_\mathfrak{q})^2$ are K-vector space.
Let $f_\mathfrak{q} : \mathfrak{m}_\mathfrak{p}/(\mathfrak{m}_\mathfrak{p})^2 \to \mathfrak{m}_\mathfrak{q}/(\mathfrak{m}_\mathfrak{q})^2$ defined by

$$f_\mathfrak{q}\left(\frac{p}{s} + (\mathfrak{m}_\mathfrak{p})^2\right) \equiv \varphi\left(\frac{p}{s}\right) + (\mathfrak{m}_\mathfrak{q})^2 \quad \text{for } p \in \mathfrak{p},\ s \in A - \mathfrak{p}.$$

If $\frac{p'}{s'} + (\mathfrak{m}_\mathfrak{p})^2 = \frac{p}{s} + (\mathfrak{m}_\mathfrak{p})^2$ for $p' \in \mathfrak{p}$ and $s' \in A - \mathfrak{p}$, then

$$\varphi\left(\frac{p}{s}\right) - \varphi\left(\frac{p'}{s'}\right) = \varphi\left(\frac{p}{s} - \frac{p'}{s'}\right) \in \varphi\left((\mathfrak{m}_\mathfrak{p})^2\right) = \varphi\left(\varphi^{-1}(\mathfrak{m}_\mathfrak{q})\right)^2 \subseteq (\mathfrak{m}_\mathfrak{q})^2,$$

so $\varphi\left(\frac{p}{s}\right) + (\mathfrak{m}_\mathfrak{q})^2 = \varphi\left(\frac{p'}{s'}\right) + (\mathfrak{m}_\mathfrak{q})^2$. $f_\mathfrak{q}$ is well-defined. Given $r = \dfrac{g\left(v^2\right)}{h\left(v^2\right)} \in K$ for $g(x) \in A,\ h(x) \in A - \mathfrak{p}$, then

$$\begin{aligned}
f_\mathfrak{q}\left(r\left(\frac{p}{s} + (\mathfrak{m}_\mathfrak{p})^2\right)\right) &= f_\mathfrak{q}\left(\left(\frac{g(x)}{h(x)} + \mathfrak{m}_\mathfrak{p}\right)\left(\frac{p}{s} + (\mathfrak{m}_\mathfrak{p})^2\right)\right) = f_\mathfrak{q}\left(\frac{g(x)\,p}{h(x)\,s} + (\mathfrak{m}_\mathfrak{p})^2\right) \\
&= \frac{g(x)\,p}{h(x)\,s} + (\mathfrak{m}_\mathfrak{q})^2 = \left(\frac{g\left(y^2\right)}{h\left(y^2\right)} + \mathfrak{m}_\mathfrak{q}\right)\left(\frac{p}{s} + (\mathfrak{m}_\mathfrak{q})^2\right) \\
&= \frac{g\left(v^2\right)}{h\left(v^2\right)}\left(\frac{p}{s} + (\mathfrak{m}_\mathfrak{q})^2\right) = r\left(\frac{p}{s} + (\mathfrak{m}_\mathfrak{q})^2\right),
\end{aligned}$$

so $f_\mathfrak{q}$ is clearly a $K-$vector space homomorphism.

(d) Suppose that $\mathfrak{q} = B(y-v)$, $\mathfrak{p} = A\left(x-v^2\right)$, and then

$$\begin{aligned}
\mathfrak{m}_{\mathfrak{p}} &= (A-\mathfrak{p})^{-1} A\left(x-v^2\right) = \left(x-v^2\right)\left((A-\mathfrak{p})^{-1} A\right) = \left(x-v^2\right) A_{\mathfrak{p}}, \\
\mathfrak{m}_{\mathfrak{q}} &= (B-\mathfrak{q})^{-1} B(y-v) = (y-v)\left((B-\mathfrak{q})^{-1} B\right) = (y-v) B_{\mathfrak{q}}, \\
(\mathfrak{m}_{\mathfrak{p}})^2 &= \left(x-v^2\right)^2 A_{\mathfrak{p}}, \quad (\mathfrak{m}_{\mathfrak{q}})^2 = (y-v)^2 B_{\mathfrak{q}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathfrak{m}_{\mathfrak{p}}/(\mathfrak{m}_{\mathfrak{p}})^2 &= \left(x-v^2\right) A_{\mathfrak{p}}/\left(x-v^2\right)^2 A_{\mathfrak{p}} = \left(x-v^2\right)\left(A_{\mathfrak{p}}/\left(x-v^2\right) A_{\mathfrak{p}}\right) \\
&= \left(x-v^2\right)\left(A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}\right) \cong K\left(x-v^2\right).
\end{aligned}$$

Similarly, $\mathfrak{m}_{\mathfrak{q}}/(\mathfrak{m}_{\mathfrak{q}})^2 \cong K(y-v)$, hence both $\mathfrak{m}_{\mathfrak{p}}/(\mathfrak{m}_{\mathfrak{p}})^2$ and $\mathfrak{m}_{\mathfrak{q}}/(\mathfrak{m}_{\mathfrak{q}})^2$ have dimension 1. Given $r = \dfrac{g\left(v^2\right)}{h\left(v^2\right)} \in K$ for $g(x) \in A,\ h(x) \in A-\mathfrak{p}$, then

$$\begin{aligned}
f_{\mathfrak{q}}\left(r\left(x-v^2\right)\right) &= f_{\mathfrak{q}}\left(\left(\frac{g(x)}{h(x)}+\mathfrak{m}_{\mathfrak{p}}\right)\left(x-v^2\right)\right) = f_{\mathfrak{q}}\left(\frac{\left(x-v^2\right) g(x)}{h(x)}+\left(x-v^2\right)\mathfrak{m}_{\mathfrak{p}}\right) \\
&= \frac{\left(y^2-v^2\right) g\left(y^2\right)}{h\left(y^2\right)}+(y-v)\mathfrak{m}_{\mathfrak{q}} = \left(\frac{(y+v) g\left(y^2\right)}{h\left(y^2\right)}+\mathfrak{m}_{\mathfrak{q}}\right)(y-v) \\
&= \left(\frac{(v+v) g\left(v^2\right)}{h\left(v^2\right)}\right)(y-v) = 2rv(y-v),
\end{aligned}$$

so $f_{\mathfrak{q}}$ isn't an isomorphism if and only if $v=0$, i.e. $\mathfrak{q} = By$.

■

7. Consider $B = K[x,y]/(xy-1)$.

   (a) Let A_1 be the subring generated by x, show that B is not integral over A_1.

   (b) Let A_2 be the subring generated by $x+y$, show that B is integral over A_2.

   (c) Show that $\dim K[x,y]/(xy-1) = 1$.

Proof.

Use the same convention as previous problem: we shall consider B as the rings $K[x,y]$ under the assumption $xy=1$. Note that here we mean $A_1 = K[x]$ and $A_2 = K[x+y]$ instead of $\mathbb{Z}[x]$ and $\mathbb{Z}[x+y]$, where $\mathbb{Z}$ is the subring of B generated by 1_K.

(a) Suppose B is integral over A_1, then y is also integral over it, say y satisfies $f(z) = z^m + \sum_{i=0}^{m-1} g_i(x) z^i \in A_1[z]$ with $g_i(x) \in A_1 = K[x]$, then

$$0 = y^m + \sum_{i=0}^{m-1} g_i(x) y^i \Rightarrow 0 = y^m x^m + \sum_{i=0}^{m-1} g_i(x) y^i x^m = 1 + \sum_{i=0}^{m-1} g_i(x) x^{m-i}.$$

This says x satisfies a nonzero polynomial over K, a clear contradiction.

(b) Note that

$$x^2 - x(x+y) + 1 = x^2 - x^2 - xy + 1 = 0,$$

so x satisfies a polynomial $z^2 - (x+y)z + 1 \in A_2[z]$. y also satisfies the same polynomial, hence both x and y are integral over A_2. In particular, y are integral over $A_2[x]$, so $A_2[x,y] = K[x,y] = B$ is integral over A_2.

(c) Because B is integral over A_2, we have $\dim B = \dim A_2$ by Going-Up theorem. That A_2 is PID implies that

$$\operatorname{Spec}(A_2) = \{A_2 f(x+y) \,|\, f : \text{irreducible polynomial over } K.\} \cup \{0\},$$

so $\dim A_2 = 1$ because each $A_2 f(x+y) \in \operatorname{Spec}(A_2)$ is maximal.

■

8. Let R be a local Noetherian domain of $\dim R = 1$. Show that R is integrally closed if and only if the maximal ideal is principal and every nonzero ideal is of the form $\mathfrak{m}^n$.

Proof.
($\Leftarrow$) Suppose $\mathfrak{m}$ is principle and every ideal is of the form $\mathfrak{m}^n$, then R is a PID and hence a UFD, so R is integrally closed.
($\Rightarrow$) We first prove that

Lemma 1 Let A be a Noetherian ring and N be a nonzero $A-$module, then $\exists x \in N - \{0\} \ni (0:x) \in \operatorname{Spec}(A)$.

Notation 2 Let A be a ring and $I \lhd A$, $J \subseteq A$. Let N be a A-module and $N_0 < N$ and $N_1 \subseteq N$. Define

$$(I:J) \equiv \{a \in A | aJ \subseteq I\}, \quad (N_0 : N_1) \equiv \{a \in A | aN_1 \subseteq N_0\}.$$

It's easy to check that $(I:J)$ and $(N_0 : N_1)$ are ideals in A.

Proof of Lemma 1.
Apply Zorn's lemma on the set $\{(0:y) \,|\, y \neq 0 \in N\}$ to get a maximal element $(0:x)$. If $bc \in (0:x)$ but $b \notin (0:x)$, then $cbx = 0$ and $bx \neq 0$. Therefore,

$$(0:x) \subseteq (0:bx) \neq A \Rightarrow (0:x) = (0:bx),$$

the implication holds from maximality of $(0:x)$. In particular, $c \in (0:bx) = (0:x)$, hence $(0:x)$ is prime. ■

R isn't a field because $\dim R \neq 0$, so $\exists x \neq 0$ is a nonunit of R. Consider R/Rx as $R/Rx-$module, then $\exists a + Rx \neq Rx \ni (Rx : a + Rx) \in \operatorname{Spec}(R/Rx)$. If R/Rx is a domain, then $Rx \neq 0 \in \operatorname{Spec}(R)$, so $\mathfrak{m} = Rx$ because $\dim R = 1$ implies there is unique nonzero prime ideal in R. Suppose R/Rx isn't a domain, then $\{\bar{0}\} \notin \operatorname{Spec}(R/Rx)$. Since the canonical projection $\pi : R \to Rx$ is surjective, $\pi^{-1}(Rx : a + Rx) \neq 0 \in \operatorname{Spec}(R)$. Again, $\dim R = 1$ implies $\mathfrak{m} = \pi^{-1}(Rx : a + Rx)$. Note that

$$\begin{aligned} r &\in (Rx : a) \Leftrightarrow ra \in Rx \Leftrightarrow \pi(r)(a + Rx) = (r + Rx)(a + Rx) = Rx \\ &\Leftrightarrow \pi(r) \in (Rx : a + Rx) \Leftrightarrow r \in \pi^{-1}(Rx : a + Rx), \end{aligned}$$

so $(Rx : a) = \pi^{-1}(Rx : a + Rx) = \mathfrak{m}$. Now let $K \equiv QF(R)$, if $\dfrac{a}{x} = b \in R$, then $a = bx$ and cancellation rule in domain R implies

$$\mathfrak{m} = (Rx : bx) = (R : b) = R, \text{contradiction!}$$

We have $\dfrac{a}{x} \in K - R$, and $\dfrac{a}{x}\mathfrak{m} \lhd R$ by $\mathfrak{m} = (Rx : a)$.

Claim 3 $\dfrac{a}{x}\mathfrak{m} \nsubseteq \mathfrak{m}$.

Lemma 4 Let A be a ring and N be a finitely generated $A-$module. If y in another extension ring of A satisfies the property: when N is assigned to be $A[y]-$module, $\operatorname{Ann}_{A[y]} N = 0$, then y is integral over A.

Proof of Lemma 4.

Let $N = \sum_{i=1}^{l} Ax_i$, when consider it as $A[y]-$module we can write $yx_j = \sum_{i=1}^{l} a_{ij}x_i$ for each j and some $a_{ij} \in A$. Therefore, rewrite it in matrix form, we get

$$(yI_l - [a_{ij}]) \begin{bmatrix} x_1 \\ \vdots \\ x_l \end{bmatrix} = 0.$$

Left multiply above formula by adjoint matrix of $yI_l - [a_{ij}]$, we get

$$\det(yI_l - [a_{ij}]) \begin{bmatrix} x_1 \\ \vdots \\ x_l \end{bmatrix} = 0 \Rightarrow \det(yI_l - [a_{ij}])\, x_i = 0 \ \forall i \Rightarrow \det(yI_l - [a_{ij}]) \in \operatorname{Ann}_{A[y]} N = 0,$$

so $\det(yI_l - [a_{ij}]) = 0$, y is integral over A. ■

Proof of Claim 3.

Now suppose that $\frac{a}{x}\mathfrak{m} \subseteq \mathfrak{m}$, then $\mathfrak{m}$ can be assigned to be $R\left[\frac{a}{x}\right]-$module. Moreover, for any $f\left(\frac{a}{x}\right) \in R\left[\frac{a}{x}\right]$ and any $m \neq 0 \in \mathfrak{m}$,

$$f\left(\frac{a}{x}\right) m = 0 \in K \Rightarrow f\left(\frac{a}{x}\right) = 0.$$

Therefore, $\operatorname{Ann}_{R[a/x]} \mathfrak{m} = 0$. $\frac{a}{x} \in QF(R)$ is integral over an integrally closed ring R, hence $\frac{a}{x} \in R$, which is a contradiction. ■

Because $\mathfrak{m}$ contains all the ideals in the local ring R, the only possibility is $\frac{a}{x}\mathfrak{m} = R \Rightarrow a\mathfrak{m} = Rx \Rightarrow \exists m \in \mathfrak{m} \ni x = am$. Again, by cancellation in domain R,

$$\mathfrak{m} = (aRm : a) = (Rm : 1_R) = Rm.$$

Now given any $I \lhd R$ but $I \notin \operatorname{Spec}(R)$, then R/I isn't a domain, that is, $\{\bar{0}\} \notin \operatorname{Spec}(R/I)$. But the canonical projection π is surjective, $\pi(\mathfrak{m}) \in \operatorname{Spec}(R/I)$, and any other $J \in \operatorname{Spec}(R/I)$ implies some $I' \in \operatorname{Spec}(R) \ni \pi(I') = J$. Therefore, $\dim R/I = 0$ and R/I is of course Noetherian, by Theorem 1.6.15 R/I is Artinian. Now consider the decreasing sequence

$$\pi(\mathfrak{m}) \supseteq \pi\left(\mathfrak{m}^2\right) \supseteq \pi\left(\mathfrak{m}^3\right) \supseteq \cdots,$$

we have $\pi(\mathfrak{m}^n) = \pi\left(\mathfrak{m}^{n+1}\right)$ for some $n \in \mathbb{N}$. Note that

$$\pi(\mathfrak{m}^n) = \pi\left(\mathfrak{m}^{n+1}\right) = \pi\left(Rm^{n+1}\right) = \pi(Rm^n)\,\pi(Rm) = \pi(\mathfrak{m}^n)\,\pi(\mathfrak{m}).$$

Since R/I is Noetherian, $\pi(\mathfrak{m}^n)$ is finitely generated $R/I-$module, then we conclude that $\pi(\mathfrak{m}^n) = 0$ by Nakayaka lemma, that is, $I = \mathfrak{m}^n$. ■