Advanced Algebra II Homework 3

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1. Determine $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$.

Proof.

Let the free $\mathbb{Z}-$module on $\mathbb{Z}_m \times \mathbb{Z}_n$ be F, and let the submodule used to create tensor product be K. Define the $\mathbb{Z}-$module homomorphism $f : F \to \mathbb{Z}_{(m,n)}$ by $f\left(\bar{a}, \bar{b}\right) = \overline{ab}$. Now for any $\bar{a}_1, \bar{a}_2, \bar{a} \in \mathbb{Z}_m,\ \bar{b}_1, \bar{b}_2, \bar{b} \in \mathbb{Z}_n$ and $r \in \mathbb{Z}$,

$$\begin{aligned}
f\left(\left(\bar{a}_1 + \bar{a}_2, b\right) - \left(\bar{a}_1, \bar{b}\right) - \left(\bar{a}_2, \bar{b}\right)\right) &= f\left(\overline{a_1 + a_2}, \bar{b}\right) - f\left(\bar{a}_1, \bar{b}\right) - f\left(\bar{a}_2, \bar{b}\right) \\
&= \overline{\left(a_1 + a_2\right) b} - \overline{a_1 b} - \overline{a_2 b} = 0, \\
f\left(\left(\bar{a}, \bar{b}_1 + \bar{b}_2\right) - \left(\bar{a}, \bar{b}_1\right) - \left(\bar{a}, \bar{b}_2\right)\right) &= f\left(\bar{a}, \overline{b_1 + b_2}\right) - f\left(\bar{a}, \bar{b}_1\right) - f\left(\bar{a}, \bar{b}_2\right) \\
&= \overline{a\left(b_1 + b_2\right)} - \overline{ab_1} - \overline{ab_2} = 0, \\
f\left(\left(\bar{a}, r\bar{b}\right) - r\left(\bar{a}, \bar{b}\right)\right) &= f\left(\bar{a}, \overline{rb}\right) - rf\left(\bar{a}, \bar{b}\right) \\
&= \overline{a\left(rb\right)} - r\overline{ab} = 0, \\
f\left(\left(r\bar{a}, \bar{b}\right) - r\left(\bar{a}, \bar{b}\right)\right) &= f\left(\overline{ra}, \bar{b}\right) - rf\left(\bar{a}, \bar{b}\right) \\
&= \overline{\left(ra\right) b} - r\overline{ab} = 0,
\end{aligned}$$

so $K \subseteq \ker f$. We may identify $F/\ker f$ as a submodule of $F/K = \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$ by $\left(\bar{a}, \bar{b}\right) + \ker f \to \left(\bar{a}, \bar{b}\right) + K = \bar{a} \otimes_{\mathbb{Z}} \bar{b}$. It's clear that $\operatorname{Im} f = \mathbb{Z}_{(m,n)}$, so $F/\ker f \cong \mathbb{Z}_{(m,n)}$, hence

$$\left|\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n\right| \geq \left|F/\ker f\right| = \left|\mathbb{Z}_{(m,n)}\right| = (m,n).$$

For any $\bar{a} \in \mathbb{Z}_m$ and $\bar{b} \in \mathbb{Z}_n$, we have

$$\bar{a} \otimes_{\mathbb{Z}} \bar{b} = ab\bar{1} \otimes_{\mathbb{Z}} \bar{1} \in \left\langle \bar{1} \otimes_{\mathbb{Z}} \bar{1} \right\rangle_{\mathbb{Z}}.$$

Since $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$ is generated from all those elements like above, we conclude that $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n = \left\langle \bar{1} \otimes_{\mathbb{Z}} \bar{1} \right\rangle_{\mathbb{Z}}$. From elementary number theory, there exists $x, y \in \mathbb{Z}$ such that $xm + yn = (m,n)$. Hence we have

$$(m,n)\, \bar{1} \otimes_{\mathbb{Z}} \bar{1} = xm\bar{1} \otimes_{\mathbb{Z}} \bar{1} + yn\bar{1} \otimes_{\mathbb{Z}} \bar{1} = x\overline{m} \otimes_{\mathbb{Z}} \bar{1} + y\bar{1} \otimes_{\mathbb{Z}} \bar{n} = 0 \Rightarrow \left|\bar{1} \otimes_{\mathbb{Z}} \bar{1}\right| \mid (m,n),$$

in particular,

$$\left|\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n\right| = \left|\bar{1} \otimes_{\mathbb{Z}} \bar{1}\right| \leq (m,n).$$

Therefore, $\left|\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n\right| = \left|\bar{1} \otimes_{\mathbb{Z}} \bar{1}\right| = (m,n)$, $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n = \left\langle \bar{1} \otimes_{\mathbb{Z}} \bar{1} \right\rangle_{\mathbb{Z}} \cong \mathbb{Z}_{(m,n).}$ $\blacksquare$

2. Let $M = M_1 \oplus M_2$. Prove that M is flat if and only if both M_i are flat. What can you say if $M = \oplus_{i \in I} M_i$ with general index set I.

Proof.

Caution When I is any index set, the symbol $\oplus_{i \in I} M_i$ actually means all the function φ on I such that $\varphi(i) \in M_i$ and $\varphi(i) = 0$ for **all but finitely** many.

Given any exact $0 \to N_1 \to N_2$, say $f_i : N_1 \otimes M_i \to N_2 \otimes M_i$, and $f : \oplus_{i \in I}\left(N_1 \otimes M_i\right) \to \oplus_{i \in I}\left(N_2 \otimes M_i\right)$ defined naturally by f_i.

Claim $\ker f = \oplus_{i \in I} \ker f_i$.

$\lceil$Given $\varphi \in \ker f$, then $f(\varphi) = 0 \Rightarrow f(\varphi)(i) = f_i(\varphi(i)) = 0$, so $\varphi(i) \in \ker f_i$ for all i. Since $\varphi \in \ker f \subseteq \oplus_{i \in I}\left(N_1 \otimes M_i\right)$, there is only finite i such that $\varphi(i) \neq 0$, hence $\varphi \in \oplus_{i \in I} \ker f_i$.

Now given $\varphi \in \oplus_{i \in I} \ker f_i$, say $0 \neq \varphi(i) \in \ker f_i$ for finite $i \in I_0$, then $f_i(\varphi(i)) = f(\varphi)(i) =$

0 for all $i \in I$, hence $\varphi \in \ker f$.$\rfloor$

By theorem IV.5.9 on Hungerford, we have

$$N \otimes M \cong \oplus_{i \in I} (N \otimes M_i)$$

for any module N. Above claim says that f_i is injective for all $i \in I$ iff f is, that is,

$$0 \to N_1 \otimes M \cong \oplus_{i \in I} (N_1 \otimes M_i) \xrightarrow{f} \oplus_{i \in I} (N_2 \otimes M_i) \cong N_2 \otimes M$$

is exact iff $0 \to N_1 \otimes M_i \xrightarrow{f_i} N_2 \otimes M_i$ is exact for all $i \in I$. Hence M is flat iff M_i is for all i. ■

3. Let R be a local ring and M is a finitely generated flat R-module. Then M is free.
In fact, if $\{x_1, ..., x_n\} \subseteq M$ such that they form a basis in $M/\mathfrak{m}M$ over $R/\mathfrak{m}$, then it forms a basis of M.

Proof.

Since M is finitely generated, so is for $M/\mathfrak{m}M$, hence $\dim_K M/\mathfrak{m}M = n < \infty$ for some $n \in \mathbb{N}$, where $K \equiv R/\mathfrak{m}$. Let $\{x_1, ..., x_n\} \subseteq M$ such that $\{\bar{x}_1, ..., \bar{x}_n\}$ forms a basis of $M/\mathfrak{m}M$, and let $\pi : R^n \to M$ defined by

$$\pi (r_1, ..., r_n) = \sum r_i x_i, \qquad r_i \in R.$$

Corollary 1.4.11 says $\langle x_1, ..., x_n \rangle = M$, so π is surjective. Do the right exact operation $\mathfrak{m}\otimes$ on the exact sequence:

$$0 \to \ker \pi \to R^n \to M \to 0$$

to get the following exact commute diagram:

$$\begin{tikzcd}
 & \mathfrak{m} \otimes \ker \pi \arrow[r] \arrow[d] & \mathfrak{m} \otimes R^n \arrow[r] \arrow[d] & \mathfrak{m} \otimes M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker \pi \arrow[r] & R^n \arrow[r] & M &
\end{tikzcd}$$

Say $f : \mathfrak{m} \otimes \ker \pi \to \ker \pi$, $g : \mathfrak{m} \otimes R^n \to R^n$ and $h : \mathfrak{m} \otimes M \to M$, Snake lemma induces the following exact sequence:

$$\ker h \to \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h).$$

Now since M is flat, exact sequence $0 \to \mathfrak{m} \to R$ induces that

$$0 \to \mathfrak{m} \otimes M \to R \otimes M \cong M$$

is exact, that is, $\ker h = 0$. Therefore,

$$0 \to \operatorname{coker}(f) = \ker \pi / \mathfrak{m} \ker \pi \to \operatorname{coker}(g) = R^n / \mathfrak{m} R^n = (R/\mathfrak{m}R)^n \to \operatorname{coker}(h) = M/\mathfrak{m}M.$$

But actually $(R/\mathfrak{m}R)^n \cong M/\mathfrak{m}M$ because we take $\{\bar{x}_1, ..., \bar{x}_n\}$ as basis of $M/\mathfrak{m}M$ over $R/\mathfrak{m}R$, so we conclude that $\ker \pi / \mathfrak{m} \ker \pi = 0$, that is, $\ker \pi = \mathfrak{m} \ker \pi$. When R is Noetherian, or M is projective, or more generally, M is finitely presented (see Lang: Lemma XVI.3.9), $\ker \pi$ is finitely generated. We can therefore apply Nakayama lemma on the relation $\ker \pi / \mathfrak{m} \ker \pi = 0$ to conclude that $\ker \pi = 0$ and $R^n \cong M$. ■

4. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules. If both M_1, M_3 are finitely generated then so is M_2.

Proof.

Let $f : M_1 \to M_2$ and $g : M_2 \to M_3$, then $\operatorname{Im} f = \ker g$ is finitely generated, say $\{x_i\}$, because M_1 is. Let $\{z_j\}$ be a generating set of M_3, surjection g permits us to choose one $y_j \in g^{-1}\{z_j\}$ for each j, consider $X = \{x_i, y_j\}$. For each $x \in M_2$, we have for some $s_j \in R$

$$g(x) = \sum_j s_j z_j = \sum_j s_j g(y_j) = g\left(\sum_j s_j y_j\right) \Rightarrow x - \sum_j s_j y_j \in \ker g,$$

hence for some $r_i \in R$,

$$x = \sum_i r_i x_i + \sum_j s_j y_j \in \langle X \rangle,$$

so M_2 is finitely generated. ■

5. * Tensor product commutes with direct limit. That is, $\varinjlim(M_i \otimes N) \cong (\varinjlim M_i) \otimes N$.

   Proof.

   See Exercise 14 to 20 in Atiyah chapter 2. ■

6. A ring S is said to be an R-algebra if there is a ring homomorphism $R \to S$. Show that S is an R-module.

   Proof.

   Let the ring homomorphism be f, define the operation $\circ : R \times S \to S$ by

   $$r \circ s \equiv f(r)\, s.$$

   For all $a, b \in R$, $x, y \in S$, we have

   $$a \circ (x + y) = f(a)(x + y) = f(a)x + f(a)y = a \circ x + a \circ y,$$
   $$(a + b) \circ x = f(a + b)x = f(a)x + f(b)x = a \circ x + b \circ x,$$
   $$a \circ (b \circ x) = a \circ (f(b)x) = f(a)f(b)x = f(ab)x = (ab) \circ x,$$
   $$1_R \circ x = f(1_R)x = 1_S x = x,$$

   so S is an $R-$module. ■

7. Let S be a flat R-algebra and M be a flat S-module. Then M is a flat R-module.

   Proof.

   We first prove the following useful lemma:

   Lemma 1 (Atiyah Exercise 2.15) Let A, B be rings, let M be an $A-$module, P a $B-$module and N an (A, B) bimodule. Then $M \otimes_A N$ is naturally a $B-$module, $N \otimes_B P$ an $A-$module, and we have

   $$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P).$$

   Proof of Lemma.

   For simply, we always use x, y, z, a, b to denote elements in M, N, P, A, B respectively. Any original product will denote without any specific symbol. $M \otimes_A N$ is an $A-$module under $A-$tensor construction, and is module$-B$ under new defined $\circ_{M \otimes_A N \times B}$:

   $$(x \otimes_A y) \circ_{M \otimes_A N \times B} b \equiv x \otimes_A (yb).$$

   Moreover, we have

   $$\begin{aligned}(a(x \otimes_A y)) \circ_{M \otimes_A N \times B} b &= ((ax) \otimes_A y) \circ_{M \otimes_A N \times B} b = (ax) \otimes_A (yb) \\ &= a(x \otimes_A (yb)) = a((x \otimes_A y) \circ_{M \otimes_A N \times B} b),\end{aligned}$$

so $M\otimes_A N$ is an (A,B) −bimodule. Similarly, $N\otimes_B P$ is an (A,B) −bimodule under $B-$tensor construction and $\circ_{A\times N\otimes_B P}$:

$$a\circ_{A\times N\otimes_B P}(y\otimes_B z)\equiv(ay)\otimes_B z.$$

$(M\otimes_A N)\otimes_B P$ is an (A,B) −bimodule under $B-$tensor construction and $\circ_{A\times(M\otimes_A N)\otimes_B P}$:

$$a\circ_{A\times(M\otimes_A N)\otimes_B P}(x\otimes_A y)\otimes_B z\equiv(a(x\otimes_A y))\otimes_B z.$$

$M\otimes_A(N\otimes_B P)$ is an (A,B) −bimodule under $A-$tensor construction and $\circ_{M\otimes_A(N\otimes_B P)}$:

$$x\otimes_A(y\otimes_B z)\circ_{M\otimes_A(N\otimes_B P)}b\equiv x\otimes_A((y\otimes_B z)b).$$

Fix an $z\in P$. The mapping $(x,y)\to x\otimes_A(y\otimes_B z)$ is clearly $A-$linear in x, and

$$x\otimes_A((ay)\otimes_B z)=x\otimes_A(a\circ_{A\times N\otimes_B P}(y\otimes_B z))=a(x\otimes_A(y\otimes_B z)),$$

where the last equality holds by $A-$linearity of $\otimes_A$. Hence $(x,y)\to x\otimes_A(y\otimes_B z)$ is $A-$linearity in both x and y, universal property induces a $A-$module homomorphism $f_z:M\otimes_A N\to M\otimes_A(N\otimes_B P)$ such that

$$f_z(x\otimes_A y)=x\otimes_A(y\otimes_B z).$$

Next, consider the mapping $(t,z)\to f_z(t)$ from $(M\otimes_A N)\times P$ into $M\otimes_A(N\otimes_B P)$,

$$\begin{aligned}f_z((x\otimes_A y)\circ_{M\otimes_A N\times B}b) &= f_z(x\otimes_A(yb))=x\otimes_A((yb)\otimes_B z)\\ &= x\otimes_A((y\otimes_B z)b)=(x\otimes_A(y\otimes_B z))\circ_{M\otimes_A(N\otimes_B P)\times B}b\\ &= f_z(x\otimes_A y)\circ_{M\otimes_A(N\otimes_B P)\times B}b,\end{aligned}$$

$$f_{bz}(x\otimes_A y)=x\otimes_A(y\otimes_B(bz))=x\otimes_A((y\otimes_B z)b)=f_z(x\otimes_A y)\circ_{M\otimes_A(N\otimes_B P)\times B}b,$$

so $(t,z)\to f_z(t)$ is $B-$linear in both t and z (summation homomorphic property is easy to verify). Universal property induces a $B-$module homomorphism $f:(M\otimes_A N)\otimes_B P\to M\otimes_A(N\otimes_B P)$ such that

$$f((x\otimes_A y)\otimes_B z)=x\otimes_A(y\otimes_B z).$$

Moreover,

$$\begin{aligned}f\left(a\circ_{A\times(M\otimes_A N)\otimes_B P}((x\otimes_A y)\otimes_B z)\right) &= f((a(x\otimes_A y))\otimes_B z)=f(((ax)\otimes_A y)\otimes_B z)\\ &= (ax)\otimes_A(y\otimes_B z)=a(x\otimes_A(y\otimes_B z))\\ &= af((x\otimes_A y)\otimes_B z),\end{aligned}$$

hence f is an (A,B) −bimodule homomorphism. Similarly, we may construct an (A,B) −bimodule homomorphism $g:M\otimes_A(N\otimes_B P)\to(M\otimes_A N)\otimes_B P$ such that

$$g(x\otimes_A(y\otimes_B z))=(x\otimes_A y)\otimes_B z.$$

Clearly $f\circ g$ and $g\circ f$ are identity maps, hence f and g are (A,B) −bimodule isomorphism. ■

Since S is a $R-$algebra, in particular, S is a (R,S) −bimodule. Now given any exact $R-$module sequence $0\to N_1\to N_2$, that S is $R-$flat implies $0\to N_1\otimes_R S\to N_2\otimes_R S$ is exact. $N_i\otimes_R S$ is nothing but extension of scalars from R to S, hence $S-$flatness of M implies $0\to(N_1\otimes_R S)\otimes_S M\to(N_2\otimes_R S)\otimes_S M$ is exact. By previous lemma and Proposition 1.5.7,

$$(N_i\otimes_R S)\otimes_S M\cong N_i\otimes_R(S\otimes_S M)\cong N_i\otimes_R M,$$

that is, $0\to N_1\otimes_R M\to N_2\otimes_R M$ is exact. Therefore, M is a flat R-module. ■

8. * Complete the exercises and incomplete proofs in the note.