

Advanced Algebra II Homework 2

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1. Let M be a finitely generated R -module. And $\varphi : M \rightarrow R^n$ is a surjective homomorphism. Show that $\ker(\varphi)$ is finitely generated.

Proof.

Let the element e_i in R^n such that all its components are zeros except the i -th component is 1, surjection property of φ permit us to choose fixed $e'_i \in \varphi^{-1}(\{e_i\})$ for all i , define $N = \langle e'_1, e'_2, \dots, e'_n \rangle$.

Claim $M = N \oplus \ker \varphi$, that is, each element in M can be uniquely written as sum of element in N and in $\ker \varphi$.

[Given any $m \in M$, then we have for some $r_1, \dots, r_n \in R$

$$\varphi(m) = (r_1, \dots, r_n) = \sum_{i=1}^n r_i e_i = \sum_{i=1}^n r_i \varphi(e'_i) = \varphi\left(\sum_{i=1}^n r_i e'_i\right),$$

so $m - \sum_{i=1}^n r_i e'_i \in \ker \varphi$, $N + \ker \varphi = M$. If $\sum_{i=1}^n r_i e'_i \in N \cap \ker \varphi$, then

$$(0, \dots, 0) = \varphi\left(\sum_{i=1}^n r_i e'_i\right) = \sum_{i=1}^n r_i \varphi(e'_i) = \sum_{i=1}^n r_i e_i = (r_1, \dots, r_n) \Rightarrow r_i = 0 \forall i,$$

so $N \cap \ker \varphi = \{0\}$.]

Now let $\{x_1, \dots, x_m\}$ be a generating set of M , say $x_i = y_i + s_i$ for $y_i \in N$, $s_i \in \ker \varphi$. Given any $s = \sum_{i=1}^m r_i x_i \in \ker \varphi$,

$$s = \sum_{i=1}^m r_i (y_i + s_i) = \sum_{i=1}^m r_i y_i + \sum_{i=1}^m r_i s_i \in N + \ker \varphi.$$

The expression in N and $\ker \varphi$ for s is unique, so $s = \sum_{i=1}^m r_i s_i$, $\ker \varphi = \langle s_1, \dots, s_m \rangle$. ■

2. Let $\mathfrak{N} \triangleleft R$ be the nilradical of R . Let S be a multiplicative set of R . Is $S^{-1}\mathfrak{N}$ the nilradical of $S^{-1}R$?

Proof.

Let the nilradical of $S^{-1}R$ be N , then $\forall x/s \in S^{-1}\mathfrak{N}$ with $x \in \mathfrak{N}$, $x^n = 0$ for some $n \in \mathbb{N}$, so $(x/s)^n = x^n/s^n = 0$ in $S^{-1}\mathfrak{N}$. Hence $S^{-1}\mathfrak{N} \subseteq N$.

Given $x/s \in N$, then $(x/s)^n = 0/1$ for some $n \in \mathbb{N}$, so $\exists s' \in S \ni s'x^n = 0$, $(s'x)^n = 0$, $s'x \in \mathfrak{N}$,

$$\frac{s'x}{s'} = \frac{x}{s} \in S^{-1}\mathfrak{N}.$$

Therefore $N = S^{-1}\mathfrak{N}$. ■

3. Let $f : M \rightarrow N$ be a R -module homomorphism. Show that f is injective if and only if for every $g, h : L \rightarrow M$ such that $fg = fh$, we have $g = h$.

Proof.

Suppose f is injective. If $f(g(x)) = f(h(x))$ for all $x \in L$, then $g(x) = h(x)$ for all $x \in L$, so $g = h$.

Suppose that $g, h : L \rightarrow M \ni fg = fh \Rightarrow g = h$, then the statement is equivalent to say $fg = 0 \Rightarrow g = 0$ because f is a R -module homomorphism. Choose $g : L \rightarrow M$, if $f(x) = 0$ for $x \in M$, then $x = g(x')$ for some $x' \in L$, so $f(g(x')) = 0$, hence $x = g(x') = 0$, f is injective. ■

4. Fix a linear transformation $A : V \rightarrow V$. We may consider V as a $K[t]$ -module. Keep the notation as in the note and assume that the minimal polynomial splits as $\prod(x - \lambda_i)^{a_i}$. Show that $V = \bigoplus V(\lambda_i)$.

Proof.

Say the minimal polynomial $p(t) = \prod(t - \lambda_i)^{a_i} \in K[t]$, then $p(A) = 0 \Rightarrow \text{Ann}(v) \neq \{0\}$ for all $v \in V$, hence V is a torsion $K[t]$ -module. Because $p(t) \in \text{Ann}(v)$ for all $v \in V$ and $q(A) = 0 \Rightarrow p(t) | q(t)$ by theorem in linear algebra, it's clear that $\text{Ann}(V) = (p(t)) = \prod((t - \lambda_i)^{a_i})$.

Claim $V = \bigoplus V(t - \lambda_i)$, where

$$\begin{aligned} V(t - \lambda_i) &= \{v \in V \mid (t - \lambda_i)^n \circ v = 0 \text{ for some } n \in \mathbb{N}\} \\ &= \{v \in V \mid (A - \lambda_i I)^n v = 0 \text{ for some } n \in \mathbb{N}\} \\ &= V(\lambda_i). \end{aligned}$$

[Define $p_i(t) = \prod_{j \neq i} (t - \lambda_j)^{a_j}$, then $\{p_i(t)\}$ are relatively prime and there exists $s_i(t) \in K[t] \ni \sum s_i(t) p_i(t) = 1_{K[t]}$. Given any $v \in V$,

$$v = 1_{K[t]} v = \sum s_i(t) p_i(t) v.$$

Since $p(t) \in \text{Ann}(V)$,

$$(t - \lambda_i)^{a_i} s_i(t) p_i(t) v = s_i(t) p(t) v = 0,$$

so $s_i(t) p_i(t) v \in V(t - \lambda_i)$. Therefore we have $V = \sum V(t - \lambda_i)$.

Let $W = \langle V(t - \lambda_j) \rangle_{j \neq i}$, suppose $v \in V(t - \lambda_i) \cap W$, then for some $n_i \in \mathbb{N}$, $(t - \lambda_i)^{n_i} v = 0$. Moreover, $v = \sum_{j \neq i} v_j$ for some $v_j \in V(t - \lambda_j)$ with $(t - \lambda_j)^{n_j} v_j = 0$ for some $n_j \in \mathbb{N}$. Therefore,

$$\left(\prod_{j \neq i} (t - \lambda_j)^{n_j} \right) v = 0.$$

Since $(t - \lambda_i)^{n_i}$ and $\prod_{j \neq i} (t - \lambda_j)^{n_j}$ are clearly relatively prime, $\exists a(t), b(t) \in K[t] \ni$

$$a(t) (t - \lambda_i)^{n_i} + b(t) \prod_{j \neq i} (t - \lambda_j)^{n_j} = 1_{K[t]}.$$

Hence

$$v = 1_{K[t]} v = a(t) (t - \lambda_i)^{n_i} v + b(t) \prod_{j \neq i} (t - \lambda_j)^{n_j} v = 0.$$

Since i is arbitrary, we complete the proof. ■

5. Let $R = K[x, y]$ and $f \in R$ be an irreducible polynomial. Let $S = \{1, f, f^2, \dots\}$. Show that S is a multiplicative set. The localization $S^{-1}R$ is usually denoted R_f . And describe $\text{Spec} R_f$ in terms of $\text{Spec} R$.

Proof.

S is clearly a multiplicative set and we know that

$$\text{Spec } S^{-1}R = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \text{Spec } R, \mathfrak{p} \cap S = \emptyset\}.$$

If $f^n \in \mathfrak{p} \cap S$, then prime property implies $f \in \mathfrak{p}$, and hence $S \subseteq \mathfrak{p}$. Therefore, either $\mathfrak{p} \supseteq S$ or $\mathfrak{p} \cap S = \emptyset$ for any $\mathfrak{p} \in \text{Spec}(R)$. But $\mathfrak{p} \supseteq S$ iff $f \in \mathfrak{p}$, so $\mathfrak{p} \cap S = \emptyset$ iff $f \notin \mathfrak{p}$. That is,

$$\text{Spec } S^{-1}R = S^{-1}(\text{Spec } R) - \{S^{-1}\mathfrak{p} \mid f \in \mathfrak{p}\},$$

where $S^{-1}(\text{Spec } R)$ means $\{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \text{Spec } R\}$. ■

6. Let $\mathfrak{p} \triangleleft R$ be a prime ideal. Then $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. Show that $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is isomorphic to the field of quotient of R/\mathfrak{p} .

Proof.

Corollary 1.4.8 says that

$$R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} = (\mathfrak{p}^c)^{-1}R/(\mathfrak{p}^c)^{-1}\mathfrak{p} \cong \pi(\mathfrak{p}^c)^{-1}R/\mathfrak{p},$$

where $\pi : R \rightarrow R/\mathfrak{p}$ is the canonical projection. It's clear that $\pi(\mathfrak{p}^c)$ consists of all nonzero elements in R/\mathfrak{p} , so $\pi(\mathfrak{p}^c)^{-1}R/\mathfrak{p}$ is the field of quotient of R/\mathfrak{p} . ■

7. Let M be a finitely generated module over a local ring (R, \mathfrak{m}) . Show that $M/\mathfrak{m}M$ can be viewed as $K \equiv R/\mathfrak{m}$ -module. And show that if $\dim_K(M/\mathfrak{m}M) = 1$, then $M = Rx$ for some $x \in M$.

Proof.

Define the operation $\circ : K \times M/\mathfrak{m}M \rightarrow M/\mathfrak{m}M$ by

$$(r + \mathfrak{m}) \circ (m + \mathfrak{m}M) = rm + \mathfrak{m}M,$$

then for any $m + \mathfrak{m}M, n + \mathfrak{m}M \in M/\mathfrak{m}M$ and any $r + \mathfrak{m}, s + \mathfrak{m} \in K$,

$$\begin{aligned} (r + \mathfrak{m}) \circ ((m + \mathfrak{m}M) + (n + \mathfrak{m}M)) &= (r + \mathfrak{m}) \circ ((m + n) + \mathfrak{m}M) \\ &= r(m + n) + \mathfrak{m}M = rm + rn + \mathfrak{m}M \\ &= (rm + \mathfrak{m}M) + (rn + \mathfrak{m}M) \\ &= (r + \mathfrak{m}) \circ (m + \mathfrak{m}M) + (r + \mathfrak{m}) \circ (n + \mathfrak{m}M), \\ ((r + \mathfrak{m}) + (s + \mathfrak{m})) \circ (m + \mathfrak{m}M) &= ((r + s) + \mathfrak{m}) \circ (m + \mathfrak{m}M) \\ &= (r + s)m + \mathfrak{m}M = rm + sm + \mathfrak{m}M \\ &= (rm + \mathfrak{m}M) + (sm + \mathfrak{m}M) \\ &= (r + \mathfrak{m}) \circ (m + \mathfrak{m}M) + (s + \mathfrak{m}) \circ (m + \mathfrak{m}M), \\ (r + \mathfrak{m}) \circ ((s + \mathfrak{m}) \circ (m + \mathfrak{m}M)) &= (r + \mathfrak{m}) \circ (sm + \mathfrak{m}M) \\ &= rsm + \mathfrak{m}M = (rs + \mathfrak{m}) \circ (m + \mathfrak{m}M) \\ &= ((r + \mathfrak{m})(s + \mathfrak{m})) \circ (m + \mathfrak{m}M), \\ (1 + \mathfrak{m}) \circ (m + \mathfrak{m}M) &= 1m + \mathfrak{m}M = m + \mathfrak{m}M. \end{aligned}$$

Therefore $M/\mathfrak{m}M$ is a K -module or a vector space over K .

Now suppose that $\dim_K(M/\mathfrak{m}M) = 1$, that is, $\exists \bar{x} \equiv x + \mathfrak{m}M \in M/\mathfrak{m}M \ni \langle \bar{x} \rangle = M/\mathfrak{m}M$, then Corollary 1.4.11 implies $\langle x \rangle = M$. ■

8. *Complete the exercises and incomplete proofs in the note.