1. Let $M$ be a finitely generated $R$-module. And $\varphi : M \to R^n$ is a surjective homomorphism. Show that $\ker(\varphi)$ is finitely generated.

**Proof.**

Let the element $e_i$ in $R^n$ such that all its components are zeros expect the $i$-th component is 1, surjection property of $\varphi$ permit us to choose fixed $e'_i \in \varphi^{-1}(\{e_i\})$ for all $i$, define $N = \langle e'_1, e'_2, \ldots, e'_n \rangle$.

**Claim** $M = N \oplus \ker \varphi$, that is, each element in $M$ can be uniquely written as sum of element in $N$ and in ker $\varphi$.

Given any $m \in M$, then we have for some $r_1, \ldots, r_n \in R$

$$\varphi(m) = (r_1, \ldots, r_n) = \sum_{i=1}^{n} r_ie_i = \sum_{i=1}^{n} r_i\varphi\left(e'_i\right) = \varphi\left(\sum_{i=1}^{n} r_ie'_i\right),$$

so $m - \sum_{i=1}^{n} r_ie'_i \in \ker \varphi, \; N + \ker \varphi = M$. If $\sum_{i=1}^{n} r_ie'_i \in N \cap \ker \varphi$, then

$$(0, \ldots, 0) = \varphi\left(\sum_{i=1}^{n} r_ie'_i\right) = \sum_{i=1}^{n} r_i\varphi\left(e'_i\right) = \sum_{i=1}^{n} r_ie_i = (r_1, \ldots, r_n) \Rightarrow r_i = 0 \forall i,$$

so $N \cap \ker \varphi = \{0\}$.

Now let $\{x_1, \ldots, x_m\}$ be a generating set of $M$, say $x_i = y_i + s_i$ for $y_i \in N$, $s_i \in \ker \varphi$. Given any $s = \sum_{i=1}^{n} r_ix_i \in \ker \varphi$,

$$s = \sum_{i=1}^{n} r_i(y_i + s_i) = \sum_{i=1}^{n} r_iy_i + \sum_{i=1}^{n} r_is_i \in N + \ker \varphi.$$

The expression in $N$ and $\ker \varphi$ for $s$ is unique, so $s = \sum_{i=1}^{n} r_is_i$, $\ker \varphi = \langle s_1, \ldots, s_m \rangle$.

2. Let $\mathfrak{m} \triangleleft R$ be the nilradical of $R$. Let $S$ be a multiplicative set of $R$. Is $S^{-1}\mathfrak{m}$ the nilradical of $S^{-1}R$?

**Proof.**

Let the nilradical of $S^{-1}R$ be $N$, then $\forall x/s \in S^{-1}\mathfrak{m}$ with $x \in \mathfrak{m}$, $x^n = 0$ for some $n \in \mathbb{N}$, so $(x/s)^n = x^n/s^n = 0$ in $S^{-1}\mathfrak{m}$. Hence $S^{-1}\mathfrak{m} \subseteq N$.

Given $x/s \in N$, then $(x/s)^n = 0/1$ for some $n \in \mathbb{N}$, so $\exists s' \in S \ni s'x^n = 0$, $(s')^n = 0$, $s'x \in \mathfrak{m}$,

$$\frac{s'x}{s's} = x/s \in S^{-1}\mathfrak{m}.$$

Therefore $N = S^{-1}\mathfrak{m}$.

3. Let $f : M \to N$ be a $R$-module homomorphism. Show that $f$ is injective if and only if for every $g, h : L \to M$ such that $fg = fh$, we have $g = h$.

**Proof.**

Suppose $f$ is injective. If $f\left(g\left(x\right)\right) = f\left(h\left(x\right)\right)$ for all $x \in L$, then $g\left(x\right) = h\left(x\right)$ for all $x \in L$, so $g = h$.

Suppose that $g, h : L \to M \ni fg = fh \Rightarrow g = h$, then the statement is equivalent to say $fg = 0 \Rightarrow g = 0$ because $f$ is a $R-$module homomorphism. Choose $g : L \to M$, if $f\left(x\right) = 0$ for $x \in M$, then $x = g\left(x'\right)$ for some $x' \in L$, so $f\left(g\left(x'\right)\right) = 0$, hence $x = g\left(x'\right) = 0$, $f$ is injective.
4. Fix a linear transformation $A : V \to V$. We may consider $V$ as a $K[t]$-module. Keep the notation as in the note and assume that the minimal polynomial splits as $\prod (x - \lambda_i)^{a_i}$. Show that $V = \bigoplus V(\lambda_i)$.

**Proof.**

Say the minimal polynomial $p(t) = \prod (t - \lambda_i)^{a_i} \in K[t]$, then $p(A) = 0 \Rightarrow \text{Ann}(v) \neq \{0\}$ for all $v \in V$, hence $V$ is a torsion $K[t]$-module. Because $p(t) \in \text{Ann}(v)$ for all $v \in V$ and $q(A) = 0 \Rightarrow p(t)|q(t)$ by theorem in linear algebra, it’s clear that $\text{Ann}(V) = (p(t)) = \prod ((t - \lambda_i)^{a_i})$.

**Claim** $V = \bigoplus V(t - \lambda_i)$, where

\[
V(t - \lambda_i) = \{v \in V | (t - \lambda_i)^n v = 0 \text{ for some } n \in \mathbb{N}\} = \{v \in V | (A - \lambda_i I)^n v = 0 \text{ for some } n \in \mathbb{N}\} = V(\lambda_i).
\]

[Define $p_i(t) = \prod_{j \neq i} (t - \lambda_j)^{a_j}$, then $\{p_i(t)\}$ are relatively prime and there exists $s_i(t) \in K[t] \ni \sum s_i(t) p_i(t) = 1_{K[t]}$. Given any $v \in V$,

\[
v = 1_{K[t]} v = \sum s_i(t) p_i(t) v.
\]

Since $p(t) \in \text{Ann}(V)$,

\[
(t - \lambda_i)^{a_i} s_i(t) p_i(t) v = s_i(t) p(t) v = 0,
\]

so $s_i(t) p_i(t) v \in V(t - \lambda_i)$. Therefore we have $V = \sum V(t - \lambda_i)$.

Let $W = (V(t - \lambda_i))_{j \neq i}$, suppose $v \in V(t - \lambda_i) \cap W$, then for some $n_i \in \mathbb{N}$, $(t - \lambda_i)^{a_i} v = 0$.

Moreover, $v = \sum_{j \neq i} v_j$ for some $v_j \in V(t - \lambda_j)$ with $(t - \lambda_j)^{a_j} v_j = 0$ for some $n_j \in \mathbb{N}$. Therefore,

\[
\left(\prod_{j \neq i} (t - \lambda_j)^{n_j}\right) v = 0.
\]

Since $(t - \lambda_i)^{a_i}$ and $\prod_{j \neq i} (t - \lambda_j)^{n_j}$ are clearly relatively prime, $\exists a(t), b(t) \in K[t]$ s.t.

\[
a(t)(t - \lambda_i)^{a_i} + b(t)\prod_{j \neq i} (t - \lambda_j)^{n_j} = 1_{K[t]}.
\]

Hence

\[
v = 1_{K[t]} v = a(t)(t - \lambda_i)^{a_i} v + b(t)\prod_{j \neq i} (t - \lambda_j)^{n_j} v = 0.
\]

Since $i$ is arbitrary, we complete the proof.]

5. Let $R = K[x, y]$ and $f \in R$ be an irreducible polynomial. Let $S = \{1, f, f^2, \ldots\}$. Show that $S$ is a multiplicative set. The localization $S^{-1}R$ is usually denotes $R_f$. And describe $\text{Spec}R_f$ in terms of $\text{Spec}R$.

**Proof.**

$S$ is clearly a multiplicative set and we know that

\[
\text{Spec} S^{-1}R = \left\{S^{-1}p | p \in \text{Spec} R, \ p \cap S = \emptyset \right\}.
\]

If $f^n \in p \cap S$, then prime property implies $f \in p$, and hence $S \subseteq p$. Therefore, either $p \supseteq S$ or $p \cap S = \emptyset$ for any $p \in \text{Spec}(R)$. But $p \supseteq S$ iff $f \in p$, so $p \cap S = \emptyset$ iff $f \notin p$. That is,

\[
\text{Spec} S^{-1}R = S^{-1}(\text{Spec} R) - \left\{S^{-1}p | f \in p \right\},
\]

where $S^{-1}(\text{Spec} R)$ means $\left\{S^{-1}p | p \in \text{Spec} R \right\}$.
6. Let \( p \triangleleft R \) be a prime ideal. Then \( R_p \) is a local ring with maximal ideal \( pR_p \). Show that \( R_p/pR_p \) isomorphic to the field of quotient of \( R/p \).

**Proof.**

Corollary 1.4.8 says that

\[
R_p/pR_p = (p^c)^{-1} R/ (p^c)^{-1} p \cong \pi (p^c)^{-1} R/p,
\]

where \( \pi : R \to R/p \) is the canonical projection. It’s clear that \( \pi (p^c) \) consists of all nonzero elements in \( R/p \), so \( \pi (p^c)^{-1} R/p \) is the field of quotient of \( R/p \). ■

7. Let \( M \) be a finitely generated module over a local ring \( (R, m) \). Show that \( M/mM \) can be viewed as \( K \)-module or a vector space over \( K \).

**Proof.**

Define the operation \( \circ : K \times M/mM \to M/mM \) by

\[
(r + m) \circ (m + mM) = rm + mM,
\]

then for any \( m + mM, n + mM \in M/mM \) and any \( r + m, s + m \in K \),

\[
(r + m) \circ ((m + mM) + (n + mM)) = (r + m) \circ ((m + n) + mM)
\]

\[
= r(m + n) + mM = rm + rn + mM
\]

\[
= (rm + mM) + (rn + mM)
\]

\[
= (r + m) \circ (m + mM) + (r + m) \circ (n + mM),
\]

\[
((r + m) + (s + m)) \circ (m + mM) = ((r + s) + m) \circ (m + mM)
\]

\[
= (r + s)m + mM = rm + sm + mM
\]

\[
= (rm + mM) + (sm + mM)
\]

\[
= (r + m) \circ (m + mM) + (s + m) \circ (m + mM),
\]

\[
(r + m) \circ ((s + m) \circ (m + mM)) = (r + m) \circ (sm + mM)
\]

\[
= rsm + mM = (rs + m) \circ (m + mM)
\]

\[
= ((r + m)(s + m)) \circ (m + mM),
\]

\[
(1 + m) \circ (m + mM) = 1m + mM = m + mM.
\]

Therefore \( M/mM \) is a \( K \)-module or a vector space over \( K \).

Now suppose that \( \dim_K (M/mM) = 1 \), that is, \( \exists x + mM \in M/mM \to \langle x \rangle = M/mM \),
then Corollary 1.4.11 implies \( \langle x \rangle = M \). ■

8. *Complete the exercises and incomplete proofs in the note.*