Advanced Algebra II Homework 1

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All the rings in this HW are commutative with identity.

1. Let $I \lhd R$ be an ideal. The radical of I is denoted $\sqrt{I}$. Prove the following:

   (a) $I \subseteq \sqrt{I}$.

   (b) $\sqrt{\sqrt{I}} = \sqrt{I}$.

   (c) $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

   (d) If $\mathfrak{p}$ is prime, then $\sqrt{\mathfrak{p}^n} = \mathfrak{p}$ for all $n > 0$.

   Proof.
   We use the equivalent definition of $\sqrt{I} = \{x \in R | x^n \in I\} = \bigcap_{\mathfrak{p}:prime, I \subseteq \mathfrak{p}} \mathfrak{p}$ by Exercise 1.1.11.

   (a) Trivial.

   (b) By (a), $\sqrt{I} \subseteq \sqrt{\sqrt{I}}$. Now given $x \in \sqrt{\sqrt{I}}$, it means that $x^n \in \sqrt{I}$ for some $n \in \mathbb{N}$, and hence $(x^n)^m = x^{nm} \in I$ for some $m \in \mathbb{N}$, so $x \in \sqrt{I}$, $\sqrt{\sqrt{I}} \subseteq \sqrt{I}$.

   (c) Let $P(I) \equiv \{\mathfrak{p} \in \operatorname{Spec}(R), I \subseteq \mathfrak{p}\}$, then definition of prime ideal implies $P(IJ) \subseteq P(I) \cup P(J)$. Since $IJ \subseteq I \cap J \subseteq I \& J$, we trivially have $P(IJ) \supseteq P(I \cap J) \supseteq P(I) \cup P(J)$, so $P(IJ) = P(I \cap J) = P(I) \cup P(J)$. Therefore,

   $$\sqrt{IJ} = \bigcap_{\mathfrak{p} \in P(IJ)} \mathfrak{p} = \sqrt{I \cap J} = \bigcap_{\mathfrak{p} \in P(I \cap J)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in P(I) \cup P(J)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in P(I)} \mathfrak{p} \cap \bigcap_{\mathfrak{p} \in P(J)} \mathfrak{p} = \sqrt{I} \cap \sqrt{J}.$$

   (d) By (a), $\mathfrak{p} \subseteq \sqrt{\mathfrak{p}}$. Given $x \in \sqrt{\mathfrak{p}}$, $x^n \in \mathfrak{p}$ for some $n \in \mathbb{N}$, so property of prime ideal implies $x \in \mathfrak{p}$, $\mathfrak{p} = \sqrt{\mathfrak{p}}$. Finally use (c) to do induction.

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2. Let $I, J \lhd R$ be ideals. Then we define the ideal quotient $(I : J) \equiv \{x \in R | xJ \subseteq I\}$. Prove the following:

   (a) $(I : J)J \subseteq I$.

   (b) $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{bc}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$.

   Proof.

   (a) For any $x \in (I : J)$ and $y \in J$, we have $xy \in I$ since $xJ \subseteq I$, so all the finite sum of such xy belong to I, that is, $(I : J)J \subseteq I$.

   (b) $x \in ((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) \Leftrightarrow x\mathfrak{c} \subseteq (\mathfrak{a} : \mathfrak{b}) \Leftrightarrow \forall c \in \mathfrak{c}, xc \in (\mathfrak{a} : \mathfrak{b}) \Leftrightarrow \forall c \in \mathfrak{c}, xc\mathfrak{b} \subseteq \mathfrak{a} \Leftrightarrow x\mathfrak{cb} \subseteq \mathfrak{a} \Leftrightarrow x \in (\mathfrak{a} : \mathfrak{cb}) = (\mathfrak{a} : \mathfrak{bc})$, so we have $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{bc})$. The same argument implies $((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b}) = (\mathfrak{a} : \mathfrak{cb})$, which is trivially the same as $(\mathfrak{a} : \mathfrak{bc})$.

   ∎

3. Every non-zero homomorphic image of a local ring is local.

Proof.

Let the unique maximal ideal be M, which contains all the ideals in R by Theorem 2.18 on Hungerford, and let f be the non-zero ring homomorphism. For any $I \lhd f(R)$, $f^{-1}(I) \lhd R$, hence $f^{-1}(I) \subseteq M$, so $I \subseteq ff^{-1}(I) \subseteq f(M) \lhd f(R)$. Clearly, $f(M)$ is the unique maximal ideal of $f(R)$, hence $f(R)$ is local. ■

4. Show that $S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$.

Proof.

Given $x/s \in S^{-1}\sqrt{I}$ for $x \in \sqrt{I}$, then $x^n \in I$ for some $n \in \mathbb{N}$, so $(x/s)^n = x^n/s^n \in S^{-1}I$, hence $x/s \in \sqrt{S^{-1}I}$. Now given $x/s \in \sqrt{S^{-1}I}$, then $(x/s)^n = x^n/s^n = y/s' \in S^{-1}I$ for some $n \in \mathbb{N}$, $y \in I$, $s' \in S$. Therefore, $\exists s'' \in S$ st. $s's''x^n = s^n s'' y \in I$, so $(s's''x)^n = (s's'')^{n-1}(s's''x^n) \in I$, $s's''x \in \sqrt{I}$, $x/s = s's''x/s's''s \in S^{-1}\sqrt{I}$. ■

5. Let $\mathfrak{a} \lhd R$ be an ideal and $\mathfrak{p}_i$ be prime ideals. Suppose that $\mathfrak{a} \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i$. Show that $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some i.

Proof.

We will do induction on n. The case of one prime ideal is trivial. Now suppose this statement holds for $n-1$ prime ideals. In the case of n prime ideals, if $\exists j$ st. $\mathfrak{a} \cap \mathfrak{p}_j \subseteq \bigcup_{i \neq j} \mathfrak{p}_i$, then $\mathfrak{a} \subseteq \bigcup_{i \neq j} \mathfrak{p}_i$ because $\mathfrak{a} \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i$ implies $\mathfrak{a} - \mathfrak{p}_i \subseteq \bigcup_{i \neq j} \mathfrak{p}_i$. Induction hypothesis says $\mathfrak{a} \subseteq \mathfrak{p}_k$ for some $k \neq j$.

Claim The case that $\mathfrak{a} \cap \mathfrak{p}_j \subsetneq \bigcup_{i \neq j} \mathfrak{p}_i$ for all j is impossible.

⌈Suppose so, then $\exists a_j \in (\mathfrak{a} \cap \mathfrak{p}_j) - \bigcup_{i \neq j} \mathfrak{p}_i$ for all j. Since $a_j \notin \bigcup_{i \neq j} \mathfrak{p}_i$, property of prime ideal implies $b \equiv a_2 a_3 \cdots a_n \notin \mathfrak{p}_1$. Furthermore, $a_j \in \mathfrak{a} \cap \mathfrak{p}_j$ implies $b \in \mathfrak{a} \cap \bigcap_{i=2}^{n} \mathfrak{p}_i$. If the element $a \equiv a_1 + b \in \bigcup_{i=1}^{n} \mathfrak{p}_i$, say $a \in \mathfrak{p}_1$, then $b = a - a_1 \in \mathfrak{p}_1$, contradict to $b \notin \mathfrak{p}_1$; say $a \in \mathfrak{p}_l$ for $l \neq 2$, then $b \in \bigcap_{i=2}^{n} \mathfrak{p}_i \subseteq \mathfrak{p}_l$ too, hence $a_1 = a - b \in \mathfrak{p}_l$, contradict to $a_1 \notin \bigcup_{i \neq 1} \mathfrak{p}_i$. Therefore, $a \in \mathfrak{a} - \bigcup_{i=1}^{n} \mathfrak{p}_i$, contradict to the condition $\mathfrak{a} \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i$.⌋ ■

6. Let R be a non-zero ring. Show that the set of prime ideals of R has minimal elements with respect to inclusion.

Proof.

Zorn's lemma implies there always exists a maximal ideal $M \lhd R$, it's easy to check that $M \in \operatorname{Spec}(R)$, hence $\operatorname{Spec}(R) \neq \emptyset$. Partially order $\operatorname{Spec}(R)$ by set inclusion. For any chain $\{\mathfrak{p}_i\}$ in $\operatorname{Spec}(R)$, clearly $\bigcap_i \mathfrak{p}_i$ is an ideal in R. Suppose there exists $ab \in \bigcap_i \mathfrak{p}_i$ but $a \notin \bigcap_i \mathfrak{p}_i$ and $b \notin \bigcap_i \mathfrak{p}_i$, then $a \notin \mathfrak{p}_j$ and $b \notin \mathfrak{p}_k$ for some j, k, say $\mathfrak{p}_j \subseteq \mathfrak{p}_k$, $b \notin \mathfrak{p}_j$. Hence $ab \notin \mathfrak{p}_j$ because $\mathfrak{p}_j$ is prime, a contradiction. Therefore, $\bigcap_i \mathfrak{p}_i \in \operatorname{Spec}(R)$, any chain in $\operatorname{Spec}(R)$ has a lower bound in $\operatorname{Spec}(R)$, $\operatorname{Spec}(R)$ has minimal elements. ■

7. *Complete the exercises in the note.