Advanced Algebra II
Mar. 23, 2007
We can consider the following "local properties"

**Proposition**

*The following are equivalent:*

1. $M = 0$.
2. $M_p = 0$ for all prime ideal $p$.
3. $M_m = 0$ for all maximal ideal $m$.

One can think of this as "a function is zero if its value at each point is zero".
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Then $Ann(x) \neq R$.
Thus $Ann(x) \subset m$ for some $m$. 
Proof.
It suffices to show that $3 \Rightarrow 1$.
Suppose that $M \neq 0$, let $x \neq 0 \in M$.
Then $Ann(x) \neq R$.
Thus $Ann(x) \subset m$ for some $m$.
Now $\frac{x}{1} = 0 \in M_m$, which means that $sx = 0$ for some $s \in (R - m) \cap Ann(x)$. This is a contradiction. \qed
Proposition

Let $\varphi : M \to N$ be an $R$-homomorphism. The following are equivalent:
1. $\varphi : M \to N$ is injective.
2. $\varphi_p : M_p \to N_p$ is injective for all prime ideal $p$.
3. $\varphi_m : M_m \to N_m$ is injective for all maximal ideal $m$. 
Proof.
Since \( \otimes R_p \) is exact, we have \( 1 \Rightarrow 2 \). \( 2 \Rightarrow 3 \) is trivial.
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It suffices to show that $3 \Rightarrow 1$.
Let $K := \ker(\varphi)$, then $K_m = \ker(\varphi_m) = 0$ since $\otimes R_m$ is exact.
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Since $\otimes R_p$ is exact, we have $1 \Rightarrow 2$. $2 \Rightarrow 3$ is trivial.
It suffices to show that $3 \Rightarrow 1$.
Let $K := \ker(\varphi)$, then $K_m = \ker(\varphi_m) = 0$ since $\otimes R_m$ is exact.
Thus we have $K = 0$ by Proposition 0.1. \qed
Proposition

The following are equivalent:
1. \( M \) is flat.
2. \( M_p \) is a flat \( R_p \)-module for all prime ideal \( p \)
3. \( M_m \) is a flat \( R_m \)-module for all maximal ideal \( m \).
Proof.
For $1 \Rightarrow 2$, if suffices to check that $M_p$ is $R_p$-flat.

Let $b \triangleleft R_p$ be an ideal, then $b = aR_p$.

Now $aR_p \otimes M_p = \left(a \otimes M\right)_p = \left(a \otimes M\right) \otimes R_p \rightarrow M \otimes R_p$, is injective because $M$ is flat and $R_p$ is flat.

To see $3 \Rightarrow 1$, for any $a \triangleleft R$, we consider $aM \rightarrow M$. Localize it, we have $(a \otimes R_M) = aR_M \otimes R_M \otimes M_m \rightarrow M_m$.

This is injective by our assumption. By Proposition 0.2, we see that $aM \rightarrow M$ is injective.
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is injective because $M$ is flat and $R_p$ is flat.
To see $3 \Rightarrow 1$,
for any $a \triangleleft R$, we consider $aM \to M$. Localize it, we have
\[ (a \otimes_R M)_m = aR_m \otimes_R M_m \to M_m. \]
Proof.
For $1 \Rightarrow 2$, if suffices to check that $M_p$ is $R_p$-flat.
Let $\mathfrak{b} \triangleleft R_p$ be an ideal, then $\mathfrak{b} = aR_p$.
Now
\[ aR_p \otimes M_p = (a \otimes M)_p = (a \otimes M) \otimes R_p \rightarrow M \otimes R_p, \]
is injective because $M$ is flat and $R_p$ is flat.
To see $3 \Rightarrow 1$,
for any $a \triangleleft R$, we consider $aM \rightarrow M$. Localize it, we have
\[ (a \otimes_R M)_m = aR_m \otimes_R M_m \rightarrow M_m. \]
This is injective by our assumption. By Proposition 0.2, we see that $aM \rightarrow M$ is injective.
In this section, we are going to survey some basic properties of Noetherian and Artinian modules.
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1. Every increasing sequence of \(x_1 \prec x_2 \prec \ldots\) in \(\Sigma\) is stationary, i.e. there exist \(n\) such that \(x_n = x_{n+1} = \ldots\).

2. Every non-empty subset of \(\Sigma\) has a maximal element.
Definition

Let $M$ be an $R$-module. Let $\Sigma$ be the set of submodules of $M$. We say that $M$ is a Noetherian (resp. Artinian) $R$-module if the P.O. set $(\Sigma, \subseteq)$ (resp. $(\Sigma, \supseteq)$) satisfies the above condition.
Remark

In the case of \((\Sigma, \subseteq)\) we say condition (1) is a.c.c. (ascending chain condition) and condition (2) is maximal condition. While in the case of \((\Sigma, \subseteq)\) we say condition (1) is d.c.c. (descending chain condition) and condition (2) is minimal condition.
Proposition

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $R$-modules. Then $M$ is Noetherian (resp. Artinian) if and only if both $M'$, $M''$ are Noetherian (resp. Artinian).
Proof.
We leave it to the readers as an exercise.
Corollary

If $M_i$ is Noetherian (resp. Artinian), then so is $\bigoplus_{i=1}^{n} M_i$. 
Proposition

$M$ is a Noetherian $R$-module if and only if every submodule of $M$ is finitely generated.
Proof.
Let $N < M$ be a submodule, let $\Sigma$ be the set of finitely generated submodules of $N$. 

By the maximal condition, there is an maximal element $N_0 \in \Sigma$. We claim that $N_0 = N$ then we are done.

To see the claim, let's suppose on the contrary that $N_0 \not\subseteq N$. Pick any $x \in N - N_0$, then $N_0 \not\subseteq N_0 + Rx < N$. And clearly, $N_0 + Rx$ is finitely generated.

This contradict to the maximality of $N_0$.

Conversely, given an ascending chain $M_1 < M_2 < \ldots$ of submodules of $M$. Let $N = \bigcup M_i$. It's clear that $N$ is a submodule of $M$. Thus $N$ is finitely generated, say $N = Rx_1 + \ldots + Rx_r$.

For each $x_i$, $x_i \in M_{j_i}$ for some $j_i$. Let $n = \max_{i = 1, \ldots, r} \{j_i\}$. Then it's easy to see that $M_n = M_{n+1} = \ldots$ and hence we are done.
Proof.
Let \( N < M \) be a submodule, let \( \Sigma \) be the set of finitely generated submodules of \( N \).
By the maximal condition, there is a maximal element \( N_0 \in \Sigma \).
We claim that \( N_0 = N \) then we are done.

To see the claim, let's suppose on the contrary that \( N_0 \nless N \).
Pick any \( x \in N - N_0 \), then \( N_0 \nless N_0 + Rx < N \).
And clearly, \( N_0 + Rx \) is finitely generated.
This contradicts the maximality of \( N_0 \).

Conversely, given an ascending chain \( M_1 < M_2 < ... \) of submodules of \( M \).
Let \( N = \bigcup M_i \).
It's clear that \( N \) is a submodule of \( M \).
Thus \( N \) is finitely generated, say \( N = Rx_1 + ... + Rx_r \).
For each \( x_i \), \( x_i \in M_{j_i} \) for some \( j_i \).
Let \( n = \max_{i=1,...,r} \{ j_i \} \).
Then it's easy to see that \( M_n = M_{n+1} = ... \) and hence we are done.
Proof.
Let $N < M$ be a submodule, let $\Sigma$ be the set of finitely generated submodules of $N$.
By the maximal condition, there is an maximal element $N_0 \in \Sigma$.
We claim that $N_0 = N$ then we are done.
To see the claim, let’s suppose on the contrary that $N_0 \nsubseteq N$.
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We claim that $N_0 = N$ then we are done.
To see the claim, let’s suppose on the contrary that $N_0 \subsetneq N$.
Pick any $x \in N - N_0$, then $N_0 \subsetneq N_0 + Rx < N$. And clearly, $N_0 + Rx$ is finitely generated.
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Let $N = \bigcup M_i$.
It’s clear that $N$ is a submodule of $M$. Thus $N$ is finitely generated, say $N = Rx_1 + \ldots + Rx_r$.
For each $x_i$, $x_i \in M_{j_i}$ for some $j_i$. Let $n = \max_{i=1,\ldots,r} \{j_i\}$. 
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We claim that $N_0 = N$ then we are done.
To see the claim, let’s suppose on the contrary that $N_0 \nless N$.
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Conversely, given an ascending chain $M_1 < M_2 < ...$ of submodules of $M$.
Let $N = \bigcup M_i$.
It’s clear that $N$ is a submodule of $M$. Thus $N$ is finitely generated, say $N = Rx_1 + ... + Rx_r$.
For each $x_i$, $x_i \in M_{j_i}$ for some $j_i$. Let $n = \max_{i=1,...,r}\{j_i\}$.
Then it’s easy to see that $M_n = M_{n+1} = ...$ and hence we are done. \qed
Definition
A ring $R$ is said to be Noetherian (resp. Artinian) if $R$ is a Noetherian (resp. Artinian) $R$-module.
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A ring $R$ is said to be Noetherian (resp. Artinian) if $R$ is a Noetherian (resp. Artinian) $R$-module. Or equivalently, the ideals of $R$ satisfies ascending (resp. descending) chain condition.
Example

1. A field if both Noetherian and Artinian.
2. The ring $\mathbb{Z}$ is Noetherian but not Artinian.
3. More generally, a PID is always Noetherian. To see this, suppose that we have $a_1 \subset a_2 \ldots$ an ascending chain of ideals. Let $a := \bigcup a_i$. Then $a$ is an ideal, hence $a = (x)$ for some $x$. Now $x \in a_n$ for some $n$, thus we have $a \subset a_n$. □

Example

Consider $R = k[x_1, x_2, \ldots]$ the polynomial ring of infinitely many indeterminate. There is an ascending chain of ideal

$$(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3) \subset \ldots$$

So $R$ is not Noetherian. Let $K$ be its quotient field, then clearly $K$ is Noetherian. Thus a subring of a Noetherian ring is not necessarily Noetherian.
However, Noetherian and Artinian properties are preserved by taking quotient.

**Proposition**

*If* $R$ *is Noetherian or Artinian, then so is* $R/\mathfrak{a}$ *for any* $\mathfrak{a} \triangleleft R$. 

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**Proof.**

Use the correspondence of ideals. We leave the detail to the readers. $\square$
However, Noetherian and Artinian properties are preserved by taking quotient.

**Proposition**

*If* $R$ *is Noetherian or Artinian, then so is* $R/a$ *for any* $a \triangleleft R$.

**Proof.**

Use the correspondence of ideals. We leave the detail to the readers.

Indeed, by using the correspondence of ideals one can also show that if $R$ is Noetherian (resp. Artinian) and then so is $S^{-1}R$. 

\[\square\]
Proposition

Let $R$ be a Noetherian (resp. Artinian) ring, and $M$ a finitely generated $R$-module. Then $M$ is Noetherian (resp. Artinian).
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Proof.
By Proposition 1.3 and 1.4.
One important result is the following:

**Theorem (Hilbert’s basis theorem)**

*If $R$ is Noetherian, then so is $R[x]$.***
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Let \( b \triangleleft R[x] \) be an ideal. We need to show that it’s finitely generated.
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Let $a$ be the set of leading coefficients of $b$, it’s easy to see that it’s an ideal. Let $a_1, ..., a_n$ be a set of generators. Then there are $f_i = a_i x^{r_i} + ... \in b$. Let $r = \max\{r_i\}$. And let $b' = (f_1, ..., f_n)$. 
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Let \( b \triangleleft R[x] \) be an ideal. We need to show that it’s finitely generated.
Let \( a \) be the set of leading coefficients of \( b \), it’s easy to see that it’s an ideal. Let \( a_1, ..., a_n \) be a set of generators. Then there are \( f_i = a_i x^{r_i} + ... \in b \). Let \( r = \max \{ r_i \} \). And let \( b' = (f_1, ..., f_n) \).
For any \( f = ax^r + ... \in b \) of degree \( r \geq m \), \( a = \sum c_ia_i \), for some \( c_i \in R \). Thus \( f - c_i x^{r - r_i} f_i \in b \) has degree \(< r \). Inductively, we get a polynomial \( g \) with degree \(< r \) and \( f = g + h \) with \( h \in b' \). 
Proof.
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Let \( a \) be the set of leading coefficients of \( b \), it’s easy to see that it’s an ideal. Let \( a_1, ..., a_n \) be a set of generators. Then there are \( f_i = a_i x^{r_i} + ... \in b \). Let \( r = \max\{r_i\} \). And let \( b' = (f_1, ..., f_n) \).
For any \( f = ax^r + ... \in b \) of degree \( r \geq m \), \( a = \sum c_i a_i \), for some \( c_i \in R \). Thus \( f - c_i x^{r-r_i} f_i \in b \) has degree \( < r \). Inductively, we get a polynomial \( g \) with degree \( < r \) and \( f = g + h \) with \( h \in b' \).
Lastly, consider \( M = R + Rx + ... + Rx^{r-1} \) a finitely generated \( R \)-module. Then \( b \cap M \) is a finitely generated \( R \)-module. So \( b = (b \cap M) + b' \) is clearly a finitely generated \( R[x] \)-module. \( \square \)
An immediate consequence is

**Corollary**

_If \( R \) is Noetherian, then so is \( R[\mathbf{x}_1, \ldots, \mathbf{x}_n] \). Also any finitely generated \( R \)-algebra is Noetherian._
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**Corollary**

*If* $R$ *is Noetherian, then so is* $R[x_1, \ldots, x_n]$. *Also any finitely generated* $R$-*algebra is Noetherian.*

By almost the same argument, one can show that $R[[x]]$ is Noetherian if $R$ is Noetherian.
We now turn into the consideration of Artinian rings.

**Proposition**

*Let R be an Artinian ring, then every prime ideal is maximal.*
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**Proof.**

Let $p$ be a prime ideal, then $B := R/p$ is an Artinian integral domain.
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**Proof.**

Let $\mathfrak{p}$ be a prime ideal, then $B := R/\mathfrak{p}$ is an Artinian integral domain.

For any $x \neq 0 \in B$, we consider $(x) \supset (x^2)$. ...
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Since $B$ is Artinian, we have $(x^n) = (x^{n+1})$ for some $n$. 
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In particular, $x^n = x^{n+1}y$. 
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Let $\mathfrak{p}$ be a prime ideal, then $B := R/\mathfrak{p}$ is an Artinian integral domain.

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Since $B$ is Artinian, we have $(x^n) = (x^{n+1})$ for some $n$.

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Hence $xy = 1$, that is $x$ is a unit.
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Thus $B$ is a field. ☐
Proposition

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Let $\Sigma$ be the set of finite intersection of maximal ideals. There is a minimal element, say $\mu := m_1 \cap \ldots \cap m_n$. It follows that $m \supset \mu \supset m_i$, and hence $m = m_i$ for some $i$. Therefore, there are $n$ maximal ideals.
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Let $\Sigma$ be the set of finite intersection of maximal ideals. There is a minimal element, say $\mu := m_1 \cap \ldots \cap m_n$. For any maximal ideal $m$, one sees that $m \cap \mu = \mu$. Therefore, there are $n$ maximal ideals.
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Let $\Sigma$ be the set of finite intersection of maximal ideals.
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$$m \supset \mu \supset m_1 \ldots m_n.$$
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Let $\Sigma$ be the set of finite intersection of maximal ideals. There is a minimal element, say $\mu := m_1 \cap \ldots \cap m_n$. For any maximal ideal $m$, one sees that $m \cap \mu = \mu$. Thus

$$m \supset \mu \supset m_1 \ldots m_n.$$ 

It follows that $m \supset m_i$, and hence $m = m_i$ for some $i$. 
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Proof.

Let $\Sigma$ be the set of finite intersection of maximal ideals. There is a minimal element, say $\mu := m_1 \cap \ldots \cap m_n$. For any maximal ideal $m$, one sees that $m \cap \mu = \mu$. Thus

$$m \supset \mu \supset m_1 \ldots m_n.$$  

It follows that $m \supset m_i$, and hence $m = m_i$ for some $i$. Therefore, there are $n$ maximal ideals.
We close this section by comparing Artinian rings and Noetherian rings.

**Theorem**

*R is Artinian if and only if R is Noetherian and dim R = 0.*
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**Theorem**

*R is Artinian if and only R is Noetherian and dim R = 0.*

Recall that \( \dim R := \sup\{n|p_0 \subsetneq p_1 \ldots p_n\} \).
We close this section by comparing Artinian rings and Noetherian rings.

**Theorem**

*R* is Artinian if and only *R* is Noetherian and \( \dim R = 0 \).

Recall that \( \dim R := \sup \{ n \mid p_0 \subsetneq p_1 \ldots p_n \} \).

The proof involves more techniques. We will prove it when we are ready.
Another important feature of Noetherian ring is that it allows us to perform the *primary decomposition*. As we’ll see later in this section that there is a nice correspondence between *algebra* (polynomial rings) and *geometry* (algebraic sets). The primary decomposition in the algebraic side coincide with the decomposition of an algebraic set into union of *irreducible components*. 
We first work on the primary decomposition for Noetherian rings.

**Definition**

An ideal $q \triangleleft R$ is said to be primary if $xy \in q$ then either $x \in q$ or $y^n \in q$ for some $n$. 
Proposition

If \( q \) is primary, then \( \sqrt{q} \) is a prime ideal. Moreover, every prime ideal containing \( q \) contains \( \sqrt{q} \).

Let \( p := \sqrt{q} \), then we say \( q \) is \( p \)-primary.

Proof.

Let \( p := \sqrt{q} \). We first show that \( p \) is prime. To this end, if \( xy \in p \), then \( x^n y^n = (xy)^n \in q \) for some \( n \). Since \( q \) is primary, one has either \( x^n \in q \) or \( (y^n)^m \in q \) for some \( m \). In any case, one has either \( x \in p \) or \( y \in p \).

Let \( p' \) be a prime ideal containing \( q \). If \( x \in p = \sqrt{q} \), then \( x^n \in q \subset p' \) for some \( n \). It follows that \( x \in p' \). Hence we have \( p \subset p' \). \qed
Definition
Let $a \triangleleft R$ be an ideal. We say that $a$ is irreducible if there is no non-trivial decomposition, i.e. for any $a = b \cap c$, then $a = b$ or $a = c$. 
Proposition

If $R$ is Noetherian, then every ideal can be written as intersection of finitely many irreducible ideals.
Proof.
Let \( a \triangleleft R \) be an ideal. Suppose that \( a \) is irreducible, then nothing to prove. If \( a \) is not irreducible, then \( a = a_1 \cap a' \). If both \( a_1 \) and \( a' \) are irreducible, then we are done. Otherwise, we may assume that \( a_1 \) is reducible. Note that \( a \subset a_1 \). By continuing this process, we get a sequence of ideals

\[
a \subset a_1 \subset a_2 \subset \ldots
\]

Since \( R \) is Noetheriann, this process must terminates and hence we are done.

Another way to put it is: Let \( \Sigma \) be the set of ideals which cannot be written as decomposition of irreducible ideals. We would like to prove that \( \Sigma \) is empty. If \( \Sigma \neq \emptyset \), by the maximal condition, there is a maximal element \( a \in \Sigma \). \( a = b \cap c \) since \( a \) is not irreducible. By the maximality of \( a \), one has that both \( b \) and \( c \) have finite decomposition, hence so is \( a \). This is the required contradiction.
Proposition

*If $R$ is a Noetherian ring, then irreducible ideal is primary.*

**Proof.**

Let $a \triangleleft R$ be an irreducible ideal. We need to show that $a$ is primary. Let’s pass to the ring $\bar{R} := R/a$. If suffices to show that if $xy = 0 \in \bar{R}$, then either $x = 0$ or $y^n = 0$ for some $n$.

**Claim.** $0 = (x) \cap (y^n)$.

Grant this claim, then by the irreducibility of $a$, one has that $0 \triangleleft \bar{R}$ is irreducible. Hence $0 = (x)$ or $0 = (y^n)$ and we are done.

To prove the claim, we consider the ascending chain

$$\text{Ann}(y) \subset \text{Ann}(y^2) \subset \ldots$$

By the a.c.c., there is $n$ such that $\text{Ann}(y^n) = \text{Ann}(y^{n+1})$. If $a \in (x) \cap (y^n)$, then $a = bx = cy^n$ for some $b, c$. $ay = 0$ since $(bx)y = b(xy) = 0$. Hence $cy^ny = cy^{n+1} = 0$. So $c \in \text{Ann}(y^{n+1}) = \text{Ann}(y^n)$. Therefore, $a = cy^n = 0$. \qed
Combining all these, we have

**Proposition**

Let $R$ be a Noetherian ring, then every ideal in $R$ has a primary decomposition, i.e. can be written as a finite union of primary ideals.

**Exercise**

Let $I \triangleleft R$ be a radical ideal in a Noetherian ring $R$, then $I = \bigcap_{i=1}^{r} \mathfrak{p}_i$ for some primes ideal $\mathfrak{p}_i$. 