Advanced Algebra II
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In this section, we are going to construct tensor product of modules coming from the multilinear algebra consideration. Then we describe its universal property. Lastly, we regard tensor product as a functor and compare its properties with the functor Hom.

Let R be a ring and M_1, M_2 be R-modules. We consider a category whose object are (f, N), where N is R-module and $f : M_1 \times M_2 \to N$ is a R-bilinear map. A morphism is defined naturally.

Definition

Keep the notation as above, the universal repelling object is called the tensor product of M_1, M_2*, denoted* $M_1 \otimes_R M_2$*.*

The existence of tensor product can be constructed as following:
Let F be the free R-module generated by the set $M_1 \times M_2$.
Let K be the submodule of F generated by R-bilinear relations, that is
$(a_1 + a_2, b) - (a_1, b) - (a_2, b)$,
$(a, b_1 + b_2) - (a, b_1) - (a, b_2)$,
$(a, rb) - r(a, b)$,
$(ra, b) - r(a, b)$.
Then we have an induced bilinear map $\varphi : M_1 \times M_2 \to F/K$.

We claim that $(\varphi, F/K)$ is the universal object.
To see this, note that for any R-bilinear map $f : M_1 \times M_2 \longrightarrow N$, one easily produce a map $h' : F \longrightarrow N$ by $h'(a,b) \mapsto f(a,b)$. Since f is bilinear, one sees that $h'(x) = 0$ if $x \in K$. Thus we have an induced map $h : F/K \longrightarrow N$.

Example

$\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3 = 0.$
$\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2 \cong \mathbb{Z}_2.$

Proposition

Let M_1, M_2, M_3 be R-modules. Then there exists a unique isomorphism $(M_1 \otimes M_2) \otimes M_3 \to M_1 \otimes (M_2 \otimes M_3)$ such that $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$.

Proof.

Fixed $z \in M_3$, we consider $f_z : M_1 \times M_2 \to M_1 \otimes (M_2 \otimes M_3)$ by $f_z(x, y) \mapsto x \otimes (y \otimes z)$. This is bilinear. By the universal property, we have $f_x : M_1 \otimes M_2 \to M_1 \otimes (M_2 \otimes M_3)$. Next consider $f : (M_1 \otimes M_2) \times M_3 \to M_1 \otimes (M_2 \otimes M_3)$ by $f(x \otimes y, z) \mapsto x \otimes (y \otimes z)$. It's easy to that this is bilinear, hence we have the required map $f : (M_1 \otimes M_2) \otimes M_3 \to M_1 \otimes (M_2 \otimes M_3)$.
The inverse map can be constructed similarly and thus we have isomorphism. □

Proposition

Let M_1, M_2 *be R-modules. There there exists a unique isomorphism* $M_1 \otimes M_2 \to M_2 \otimes M_1$ *such that* $x \otimes y \mapsto y \otimes x$*.*

Proposition

Let M_R, M'_R *be right R-modules and* ${}_RN, {}_RN'$ *be left R-modules. And let* $f : M \to M'$, $g : N \to N'$ *be module homomorphisms. Then there is a unique group homomorphism* $f \otimes g : M \otimes_R N \to M' \otimes_R N'$*.*

Proof.

Consider a middle linear map $\overline{(f, g)} : M \times N \to M' \otimes_R N$ by $(a, b) \mapsto f(a) \otimes g(b)$. By the universal property, we are done. □

There are some more properties:

Proposition

$$(\oplus_{i=1}^{n} M_i) \otimes N \cong \oplus_{i=1}^{n} (M_1 \otimes M).$$

In fact, this also holds if the index set in infinite.

Also we have

Proposition

$M \otimes_R R \cong M$

Proof.

There is a natural map $\jmath : N \to R \otimes_R N$ by $\jmath(x) = x \otimes 1$. It's clear that this is an R-homomorphism. We then construct $f : R \times N \to N$ by $f(r, x) = rx$. It's clear that this is middle linear and thus induces a group homomorphism $\bar{f} : R \otimes_R N \to N$ by $\bar{f}(r \otimes x) = rx$. It's also easy to see that this is a module homomorphism.

Therefore, it suffices to check that $\bar{f}\jmath = \mathbf{1}_N$ (which is clear) and $\jmath\bar{f} = \mathbf{1}_{R \otimes_R N}$. This mainly due to

$$\sum r_i \otimes x_i = \sum (1 \otimes r_i x_i) = 1 \otimes \sum r_i x_i.$$

□

Combining these two, we have

Proposition

If F is free over R with basis $\{v_i\}_{i\in I}$. Then every element of $M\otimes_R F$ can be written as $\sum_{i\in I} x_i \otimes v_i$, with $x_i \in M$ and all but finitely many $x_i = 0$.

Moreover,

Proposition

If M, N are free over R with basis $\{v_i\}, \{w_i\}$ respectively. Then $M\otimes_R N$ is free with basis $\{v_i \otimes w_j\}$.

We now consider the "base change".
That is, if $f : R \to S$ is a ring homomorphism.
Then there are connection between S-modules and R-modules.
First, if N is a S-module, then N can be viewed as an R-module by $R \times N \to N$ such that $(r, x) \mapsto f(r)x$. This operation is called restriction of scalars.
For example, a vector space V over $\mathbb{Q}$ can be viewed as a $\mathbb{Z}$-module.
On the other hand, suppose now that we have M a R-module. S can be viewed as R-module. So we have $M_S := S \otimes_R M$, which is naturally a S-module. This operation is called base change.
For example, let $M = \mathbb{Z}[x]$ be a $\mathbb{Z}$-module and $S = \mathbb{Q}$, then $M_S = \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[x]$.

Exercise

Let S be a multiplicative set in R. Then we have $\imath : R \to S^{-1}R$. Let M be an R-module, then $S^{-1}M \cong S^{-1}R \otimes_R M$. $\square$

Exercise

Show that $S^{-1}(M \otimes_R N) \cong S^{-1}M \otimes_{s^{-1}R} S^{-1}N$. In particular, we have $(M \otimes N)_{\mathfrak{p}} \cong M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$.

Proposition

Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ *be an exact sequence of R-modules. And N is an R-module. Then* $M_1 \otimes N \longrightarrow M_2 \otimes N \longrightarrow M_3 \otimes N \longrightarrow 0$ *is exact. That is, tensor product ir right exact.*

Proof.

For $y \in N_3$, $y = g(z)$ for some $z \in N_2$, thus for $x \in M$, $x \otimes y = (\mathbf{1} \otimes g)(x \otimes z)$. Hence $\mathrm{im}(\mathbf{1} \otimes g)$ generate $M \otimes_R N_3$. It follows that $\mathbf{1} \otimes g$ is surjective.

$$(\mathbf{1} \otimes g)(\mathbf{1} \otimes f)(x \otimes w) = x \otimes gf(w) = x \otimes 0 = 0.$$

Therefore, $\mathrm{im}(\mathbf{1} \otimes f) \subset \ker(\mathbf{1} \otimes g)$. There is thus an induced map $\alpha : M \otimes_R N_2 / \mathrm{im}(\mathbf{1} \otimes f) \to M \otimes_R N_3$. It suffices to show that α is an isomorphism. To this end, we intend to construct the inverse map. Consider $x \otimes y \in M \otimes_R N_3$, there is $z \in N_2$ such that $g(z) = y$. We define $\beta_0 : M \times N_3 \to M \otimes_R N_2 / \mathrm{im}(\mathbf{1} \otimes f)$ by $\beta_0(x, y) = \overline{x \otimes z}$. We first check that this is well-defined. If $z, z' \in N_2$ such that $g(z) = g(z') = y$, then $z - z' \in \ker g = \mathrm{im} f$. Thus there is $w \in N_1$ such that $z - z' = f(w)$. One verifies that

$$\overline{x \otimes z} = \overline{x \otimes (z' + f(w))} = \overline{x \otimes z'} + \overline{x \otimes f(w)}$$

$$= \overline{x \otimes z'} + \overline{(\mathbf{1} \otimes f)(x \otimes w)} = \overline{x \otimes z'}.$$

It's routine to check that β_0 is middle linear, hence it induces

Another way to see it is via the relation with Hom functor.

Lemma

The sequence $M_1 \to M_2 \to M_3 \to 0$ *is exact if and only if* $0 \to \mathrm{Hom}(M_3, N) \to \mathrm{Hom}(M_2, N) \to \mathrm{Hom}(M_1, N)$ *is exact for all* N*.*

Lemma

There is a canonical isomorphism $\mathrm{Hom}(M \otimes N, P) \cong \mathrm{Hom}(M, \mathrm{Hom}(N, P))$*.*

Proof of Prop. 0.12.

Since $M_1 \to M_2 \to M_3 \to 0$ is exact,
we have for all P,

$$0 \to \mathrm{Hom}(M_3, \mathrm{Hom}(N, P)) \to \mathrm{Hom}(M_2, \mathrm{Hom}(N, P))$$

$$\to \mathrm{Hom}(M_1, \mathrm{Hom}(N, P)),$$

is exact.
Thus

$$0 \to \mathrm{Hom}(M_3 \otimes N, P) \to \mathrm{Hom}(M_2 \otimes N, P) \to \mathrm{Hom}(M_1 \otimes N, P),$$

is exact for all P.
And hence $M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0$ is exact. □

This says that the functor $\otimes_R N$ is right exact. Similarly, one can see that the functor $N \otimes_R$ is also right exact.

Definition

A module is said to be **flat** *if the functor* $\otimes_R M$ *is exact.*

For example, $S^{-1}R$ is a flat R-module.
We have the following easier criterion for flatness.

Proposition

The following are equivalent:

1. N *is flat.*
2. *If* $M_1 \to M_2$ *is injective, then* $M_1 \otimes N \to M_2 \otimes N$ *is injective.*

The criterion is not so effective so far. We can have an effective one

Theorem

M *is flat if and only if* $\otimes_M$ *is exact with respect to* $0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$ *for all ideal* $\mathfrak{a}$.

Then it follows that for example, $S^{-1}R$ is a flat R-module.
$R/\mathfrak{a} \otimes M \cong M/\mathfrak{a}M$.

Proof.

1. $R/\mathfrak{a} \otimes M \cong M/\mathfrak{a}M$.

2. We introduce the notion of N-flat if $\otimes M$ is exact for any N' such that $0 \to N' \to N$.

3. If M is N-flat, then M is $\oplus N$-flat.

4. If M is N-flat, then also for every submodule and quotient of N.

5. Since every module is quotient of free modules, we are done. □

To see the step 4,

$0 \to S \to N \to Q \to 0$. Let $0 \to Q' \to Q$, and N' be its preimage in N.

We have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & N' & \longrightarrow & Q' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S & \longrightarrow & N & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

Tensoring with M, we get

$$\begin{array}{ccccccccc} & & & & 0 & & K & & \\ & & & & \downarrow & & \downarrow & & \\ & & S \otimes M & \longrightarrow & N' \otimes M & \longrightarrow & Q' \otimes M & \longrightarrow & 0 \\ & & = \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S \otimes M & \longrightarrow & N \otimes M & \longrightarrow & Q \otimes M & & \end{array}$$

By Snake Lemma, we have $K = 0$.

We can consider the following "local properties"

Proposition

The following are equivalent:

1. $M = 0$.
2. $M_{\mathfrak{p}} = 0$ *for all prime ideal* $\mathfrak{p}$
3. $M_{\mathfrak{m}} = 0$ *for all maximal ideal* $\mathfrak{m}$.

Proposition

The following are equivalent:

1. $f : M \to N$ *is injective.*
2. $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ *is injective for all prime ideal* $\mathfrak{p}$
3. $f : M_{\mathfrak{m}} \to N_{\mathfrak{p}}$ *is injective for all maximal ideal* $\mathfrak{m}$.

Proposition

The following are equivalent:

1. M *is flat.*
2. $M_{\mathfrak{p}}$ *is flat for all prime ideal* $\mathfrak{p}$
3. $M_{\mathfrak{m}}$ *is flat for all maximal ideal* $\mathfrak{m}$.