Advanced Algebra II
May. 11, 2007
We will first discuss the singularities of affine varieties. Then we introduce the notion of Zariski tangent space and realize the singularities as a local property.
Definition

Let $Y \subset \mathbb{A}^n$ be an affine variety, and let $f_1, \ldots, f_t$ be a set of generators of the ideal of $Y$. We say that $Y$ is non-singular at $p$ if the rank of the Jacobian matrix at $p$, $(\frac{\partial f_i}{\partial x_j}(p))$ is $n - r$, where $r = \dim Y$. 
Note that if $p$ is a smooth point, then the tangent space at $p$ is nothing but the subspace defined by $\sum_j \frac{\partial f_i}{\partial x_j}(p)(x_j - p_j) = 0$. 
Note that if $p$ is a smooth point, then the tangent space at $p$ is nothing but the subspace defined by $\sum_j \frac{\partial f_i}{\partial x_j}(p)(x_j - p_j) = 0$. So the definition matches the usual definition of tangent spaces if the point is smooth.
Definition

Let \(((R, \mathfrak{m}))\) be a Noetherian local \(k\)-algebra and \(R/\mathfrak{m} = k\). Then we say \(R\) is regular if \(\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A\).
Theorem

Let \( Y \subset \mathbb{A}^n \) be an affine variety and \( p \in Y \) a point. Then \( Y \) is non-singular at \( p \) if and only if \( \mathcal{O}_p \) is regular.
Proof.
Let \( I = \mathcal{I}(Y) \) and \( \mathfrak{n} \triangleleft k[x_1, \ldots, x_n] \) the maximal ideal corresponding to \( p \). Clearly \( I \subset \mathfrak{n} \).
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We define a \( k \)-linear map \( \phi : k[x_1, ..., x_n] \rightarrow k^n \) by

\[
\phi(f) := (\frac{\partial f}{\partial x_1}(p), \frac{\partial f}{\partial x_2}(p), ..., \frac{\partial f}{\partial x_n}(p)).
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\( \phi \) induces an isomorphism \( \psi : \mathfrak{n}/\mathfrak{n}^2 \rightarrow k^n \).
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The rank of Jacobian is nothing but the dimension of image of \( \phi(I) = (I + \mathfrak{n}^2)/\mathfrak{n}^2 \).
Proof.
Let \( I = \mathcal{I}(Y) \) and \( n \subset k[x_1, \ldots, x_n] \) the maximal ideal corresponding to \( p \). Clearly \( I \subset n \).
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\( \phi \) induces an isomorphism \( \psi : n/n^2 \to k^n \).
The rank of Jacobian is nothing but the dimension of image of \( \phi(I) = (I + n^2)/n^2 \).
On the other hand, we have \( m = n/I \), \( m^2 = (I + n^2)/I \). Thus \( m/m^2 \cong n/(I + n^2) \).
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On the other hand, we have $m = \mathfrak{n}/I$, $m^2 = (I + \mathfrak{n}^2)/I$. Thus $m/m^2 \cong \mathfrak{n}/(I + \mathfrak{n}^2)$.
It follows that we have

$$
\dim m/m^2 + \text{rank}(J) = n.
$$

And we are done. \qed
In fact one can show that

**Exercise**

Let \((R, \mathfrak{m})\) be a Noetherian local \(k\)-algebra and \(R/\mathfrak{m} = k\). Then \(\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim A\).
Theorem

Let $Y$ be a variety. Then the set of singularities $\text{Sing}(Y)$ is a proper closed subset of $Y$. 
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Suppose that $Y$ is a hypersurface in $\mathbb{A}^n$. If $\text{Sing}(Y) = Y$, then $\frac{\partial f}{\partial x_i}(p) = 0$ for all $i$, and for all $p$. Thus $\frac{\partial f}{\partial x_i} = 0$ for all $i$. If $\text{char} k = 0$, then this can not happen. If $\text{char} k = p > 0$, then this happened only when $f = g^p$ for some $g$. Then $f$ is not irreducible, a contradiction.
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Example (Resolution of singularities via blowing-ups)

Let \( Y := \mathcal{V}(y^2 - x^3) \subset \mathbb{A}^2 \). It has a singularity at \((0, 0)\). We can resolve the singularity via blowups.
Theorem

Let $F, G \in k[x, y, z]$ be two homogeneous polynomial of degree $n, m$ respectively with no common component. Then

$$\sum_{P \in F \cap G} I_P(F, G) = mn,$$

where $I_P(F, G)$ denotes the intersection multiplicities.
Let $S = k[x, y, z] = \bigoplus S_d$ be the (graded) homogeneous coordinate ring of $\mathbb{P}^2$. Let $M = \bigoplus M_d$ be a graded $S$-module, i.e. $S_d M_e \subset M_{d+e}$. We define the Hilbert function as $h_M(l) := \dim_k M_l$. It’s easy to see that $h_S(l) = C_2^{l+2}$. 
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$h_M(l) := \dim_k M_l$. It’s easy to see that $h_S(l) = C_2^{l+2}$.
Let $M$ be a graded module, we define $M(r)$ to be the shifted graded module by $M(r)_d := M_{d+r}$. Then for any homogeneous polynomial $F$ of degree $m$, we have an exact sequence of graded modules:

$$0 \rightarrow S(−m) \rightarrow S \rightarrow S/F \rightarrow 0.$$
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$$0 \rightarrow S(-m) \rightarrow S \rightarrow S/(F) \rightarrow 0.$$

It’s then clear that

$$h_F(l) := h_S(l) - h_S(l - m) = ml + (m^2 + 3m)/2.$$
Now given a homogeneous polynomial $G$ of degree $n$ with no common factors with $F$. We then consider

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Let $M := S/(F, G)$. There is a filtration $0 = M_0 \subset M_1 \subset M_r = M$ such that $M_i/M_{i-1} \cong R/p_i$ for some $p_i$. 

In fact, $p_i$ appears if and only if $p_i = mP$ for some $P \in V(F, G)$. It appears $t$-times if and only if $\text{length } O_P/(f, g) = t$, which is the intersection multiplicities. By computing the Hilbert polynomial, we prove the Bezout’s theorem.
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In fact, $p_i$ appears if and only $p_i = m_P$ for some $P \in \mathcal{V}(F, G)$. It appears $t$-times if and only if $\text{length} \mathcal{O}_P/(f, g) = t$, which is the intersection multiplicities.
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By computing the Hilbert polynomial. We prove the Bezout's theorem.
There are some classical projective geometry which can be realized as corollaries of this theorem:

**Corollary (Desargues theorem)**

**Corollary (Pascal mystic hexagon)**
Moreover, one can define a group structure on a non-singular cubic curve $F$.
For a given non-singular curve $F$, we first define a Hessian matrix

$$
\mathcal{H} = \det \begin{pmatrix}
F_{xx} & F_{xy} & F_{xz} \\
F_{yx} & F_{yy} & F_{yz} \\
F_{zx} & F_{zy} & F_{zz}
\end{pmatrix}.
$$

Recall that for a homogeneous polynomial $P$ of degree $d$, we have

$$dP = xP_x + yP_y + zP_z.$$
Exercise

A point $p$ on $\mathcal{V}(F)$ is called an inflection point if $\mathcal{H}(p) = 0$. Show that this is equivalent to $I_P(L, F) \geq 3$, where $L$ denotes the tangent line of $\mathcal{V}(F)$ at $p$.

Also shows that $H$ and $F$ have no common component.
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if $P, Q, R$ are the three intersection point of a line $L$ with $F$, then $P \oplus Q \oplus R = o$. 
Remark

An 1-dimensional regular local ring is a DVR. (cf. [Matsumura, p.79])
Let $X$ be a variety (maybe singular), a divisor $D = \sum a_iD_i$ is a finite formal sum of irreducible subvarieties of codimension 1. One can define $\text{Div}(X)$ to be the divisor group. It’s nothing but the free abelian group on the set of irreducible subvarieties of codimension 1.
Let $K(X)$ be the field of rational functions. If $X$ is non-singular in codimension 1, (i.e. localization at height 1 prime gives a regular local ring.)
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by $\text{div}(f) = \sum_Y v(Y)$. Where the sum is taking over all codimension 1 subvarieties. The image is called principal divisors. Two divisor are said to be linearly equivalent if they differ by a principal divisor. We define the divisor class group $\text{Cl}(X) := \text{Div} / \{\text{principal divisors}\}$. 
Exercise

Prove that $Cl(\mathbb{P}^n) \cong \mathbb{Z}$. Hence we can define the degree of a divisor via $\text{Div}(\mathbb{P}^n) \to Cl(\mathbb{P}^n) \to \mathbb{Z}$. 
However, the divisor class group is in general not that simple. For example, $Cl(E) \not\cong \mathbb{Z}$.

Every divisor on a non-singular variety is locally principal, i.e. in a sufficiently small neighborhood $U_\alpha$, $D|_{U_\alpha} = div(f_\alpha)$ for some rational function $f_\alpha$. 

We call $f_\alpha$ the local equation of $D$. Note that on $U_\alpha \cap U_\beta$, $f_\alpha f_\beta^{-1}$ is regular. On the other hand, if one has an open covering $X = \bigcup U_\alpha$ and a collection of $(f_\alpha, U_\alpha)$ such that $f_\alpha f_\beta^{-1}$ is regular, then this defines a divisor.
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Example

Consider \( \mathbb{P}^1 = U_0 \cup U_1 \). Let \( t, s \) be local coordinate of \( U_0, U_1 \) respectively. One has \( s = t^{-1} \). Also \( K(X) = k(t) = k(s) \). Now consider 1-form \( dt \) on \( U_0 \), it’s clear that \( dt = -ds/s^2 \). We have \( \{(1, U_0), (-1/s^2, U_1)\} \) which represent the 1-form. The divisor is \(-2[\infty]\) which is the canonical divisor.
Example

Similarly, consider $X = \mathbb{P}^n = U_0 \cup \ldots \cup U_n$. Computation shows that $K_X = -(n + 1)H$ for some hyperplane $H$. 
Remark

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different divisor. Indeed, they might give different divisor but still
linear equivalent.

One should say that the canonical divisor is the equivalent class of
the divisor defined this way. Or sometimes we simply said that a
divisor is a canonical divisor if it is in the linear equivalent class.
For a given a divisor $D = \sum n_i D_i$, we say $D$ is effective, denoted $D \geq 0$, if $n_i \geq 0$ for all $i$. The linear series of $D$ is defined as

$$|D| := \{D' \in \text{Div}(X) | D' \sim D, D' \geq 0\}.$$