Advanced Algebra II

Homework 4

due on Apr. 20, 2007

(1) * Complete the exercises and incomplete proofs in the note.
(2) Complete the proof of Noether normalization theorem and Hilbert's Nullstellensatz.
(3) Let G be a finite group of automorphism of R, and let R^G be the subring of G-invariants, i.e. $R^G := \{x \in R | \sigma(x) = x, \forall \sigma\}$. Show that R is integral over R^G.
(4) Let k be an algebraically closed field. For any non-constant $f \in k[x_1, ..., x_n]$. Show that $\mathcal{V}(f)$ is an infinite set.
(5) Keep the notation as above, let $R = k[x, y]$ and $G \cong \mathbb{Z}_n$ acting on R by the generator $\sigma(x) \mapsto \zeta x, \sigma(y) \mapsto \zeta^{-1} y$, where $\zeta = e^{\frac{2\pi i}{n}}$. Determine R^G.
(6) Given a ring R, one can consider $\mathrm{Spec}(R)$. Moreover, one can define the Zariski topology on the set $\mathrm{Spec}(R)$ by the following: For any $\mathfrak{a} \in R$, we define $\mathcal{V}(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{a} \subset \mathfrak{p}\}$.
Show that this gives a topology.
When $R = k[x, y]$ with k algebraically closed. Compare the Zariski topology on $\mathbb{A}^2_k$ and the one on $k[x, y]$.