Advanced Algebra II

Homework 4

due on Apr. 13, 2007

(1) * Complete the exercises and incomplete proofs in the note.

(2) Let M be a Noetherian R-module, and let $\mathfrak{a} \lhd$ be the annihilator of M. Prove that M is a Noetherian $R/\mathfrak{a}$-module. How about if we replace Noetherian by Artinian?

(3) * Let R be a Noetherian local ring and M be a finitely generated R-module. Show that M is free if and only if M is flat.

(4) Let R be an Noetherian ring, and $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal. Show that there exists $n \geq 1$ such that $\mathfrak{p}^n \subset \mathfrak{q}$.
Is it still true if R is not necessarily Noetherian?

(5) Let $f : A \to B$ be a homomorphism between local rings $(A, \mathfrak{m}_A), (B, \mathfrak{m}_B)$. We say that f is **local** if $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.
If we start with a homomorphism $f : A \to B$ of rings. For any $\mathfrak{q} \in \operatorname{Spec}(B)$, we have $\mathfrak{p} := f^{-1}\mathfrak{q} \in \operatorname{Spec}(A)$. Show that the induced map $A_\mathfrak{p} \to B_\mathfrak{q}$ is local.

(6) Let k be an algebraically closed field. Consider the ring homomorphism $f : A := k[x] \to B := k[x, y]/(y^2 - x)$ which sends $f(x) = x$.
   (a) Show that B is integral over A.
   (b) For each prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$, determine the prime ideals of B lying over $\mathfrak{p}$.
   (c) Show that for each prime ideal $\mathfrak{q} \in \operatorname{Spec}(B)$, lying over $\mathfrak{p}$, we have a local homomorphism $(A_\mathfrak{p}, \mathfrak{m}_\mathfrak{p}) \to (B_\mathfrak{q}, \mathfrak{m}_\mathfrak{q})$. Moreover, a k-vector space homomorphism $f_\mathfrak{q} : \mathfrak{m}_\mathfrak{p}/(\mathfrak{m}_\mathfrak{p})^2 \to \mathfrak{m}_\mathfrak{q}/(\mathfrak{m}_\mathfrak{q})^2$.
   (d) Show that for $\mathfrak{q} \neq 0$, all the above vector space $\mathfrak{m}_\mathfrak{p}/(\mathfrak{m}_\mathfrak{p})^2, \mathfrak{m}_\mathfrak{q}/(\mathfrak{m}_\mathfrak{q})^2$ has dimension 1. And also determine when $f_\mathfrak{q}$ is not isomorphism.

(7) Consider $B = k[x, y]/(xy - 1)$.
   (a) Let A_1 be the subring generated by x, show that B is not integral over A_1.
   (b) Let A_2 be the subring generated by $x + y$, show that B is integral over A_2.
   (c) Show that $\dim k[x, y]/(xy - 1) = 1$.

(8) Let R be a local Noetherian domain of $\dim R = 1$. Show that R is integrally closed if and only if the maximal ideal is principal and every ideal is of the form $\mathfrak{m}^n$.