(1) * Complete the exercises and incomplete proofs in the note.
(2) Let $M$ be a Noetherian $R$-module, and let $\mathfrak{a} \triangleleft M$ be the annihilator of $M$. Prove that $M$ is a Noetherian $R/\mathfrak{a}$-module. How about if we replace Noetherian by Artinian?
(3) * Let $R$ be a Noetherian local ring and $M$ be a finitely generated $R$-module. Show that $M$ is a Noetherian $R/\mathfrak{a}$-module. How about if we replace Noetherian by Artinian?
(4) Let $R$ be a Noetherian local ring and $M$ be a finitely generated $R$-module. Show that $M$ is free if and only if $M$ is flat.
(5) Let $R$ be an Noetherian ring, and $q$ be a $p$-primary ideal. Show that there exists $n \geq 1$ such that $p^n \subset q$.
Is it still true if $R$ is not necessarily Noetherian?
(6) Let $k$ be an algebraically closed field. Consider the ring homomorphism $f: A := k[x] \rightarrow B := k[x, y]/(y^2 - x)$ which sends $f(x) = x$.
   (a) Show that $B$ is integral over $A$.
   (b) For each prime ideal $\mathfrak{p} \in \text{Spec}(A)$, determine the prime ideals of $B$ lying over $\mathfrak{p}$.
   (c) Show that for each prime ideal $\mathfrak{q} \in \text{Spec}(B)$, lying over $\mathfrak{p}$, we have a local homomorphism $(A_\mathfrak{p}, \mathfrak{m}_\mathfrak{p}) \rightarrow (B_\mathfrak{q}, \mathfrak{m}_\mathfrak{q})$. Moreover, a $k$-vector space homomorphism $f_\mathfrak{q}: \mathfrak{m}_\mathfrak{p}/(\mathfrak{m}_\mathfrak{p})^2 \rightarrow \mathfrak{m}_\mathfrak{q}/(\mathfrak{m}_\mathfrak{q})^2$.
   (d) Show that for $\mathfrak{q} \neq 0$, all the above vector space $\mathfrak{m}_\mathfrak{p}/(\mathfrak{m}_\mathfrak{p})^2, \mathfrak{m}_\mathfrak{q}/(\mathfrak{m}_\mathfrak{q})^2$ has dimension 1. And also determine when $f_\mathfrak{q}$ is not isomorphism.
(7) Consider $B = k[x, y]/(xy - 1)$.
   (a) Let $A_1$ be the subring generated by $x$, show that $B$ is not integral over $A_1$.
   (b) Let $A_2$ be the subring generated by $x + y$, show that $B$ is integral over $A_2$.
   (c) Show that $\dim k[x, y]/(xy - 1) = 1$.
(8) Let $R$ be a local Noetherian domain of $\dim R = 1$. Show that $R$ is integrally closed if and only if the maximal ideal is principal and every ideal is of the form $\mathfrak{m}^n$. 