Advanced Algebra II

Homework 2

due on Mar. 16, 2007

(1) Let M be a finitely generated R module. And $\varphi : M \to R^n$ is a surjective homomorphism. Show that $\ker(\varphi)$ is finitely generated.
(2) Let $\mathfrak{N} \lhd R$ be the nilradical of R. Let S be a multiplicative set of R. Is $S^{-1}\mathfrak{N}$ the nilradical of $S^{-1}R$?
(3) Let $f : M \to N$ be a R-module homomorphism. Show that f is injective if and only if for every $g, h : L \to M$ such that $fg = fh$, we have $g = h$.
(4) Fix a linear transformation $A : V \to V$. We may consider V as a $k[t]$-module. Keep the notation as in the note and assume that the minimal polynomial splits as $\prod(x - \lambda_i)^{a_i}$. Show that $V = \oplus V(\lambda_i)$.
(5) Let $R = k[x, y]$ and $f \in R$ be an irreducible polynomial. Let $S = \{1, f, f^2, ...\}$. Show that S is a multiplicative set. The localization $S^{-1}R$ is usually denotes R_f. And describe $\mathrm{Spec} R_f$ in terms of $\mathrm{Spec} R$.
(6) Let $\mathfrak{p} \lhd R$ be a prime ideal. Then $R_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Show that $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ isomorphic to the field of quotient of $R/\mathfrak{p}$.
(7) Let M be a finitely generated module over a local ring $(R, \mathfrak{m})$. Show that $M/\mathfrak{m}M$ can be viewed as $k := R/\mathfrak{m}$-module. And show that if $\dim_k M/\mathfrak{m}M = 1$, then $M = Rx$ for some $x \in M$.
(8) * Complete the exercises and incomplete proofs in the note.