Advanced Algebra II

Homework 1

due on Mar. 9, 2007

(1) Let $I \lhd R$ be an ideal. The radical of I is denoted $\sqrt{I}$. Prove the following:
   (a) $I \subset \sqrt{I}$.
   (b) $\sqrt{\sqrt{I}} = \sqrt{I}$.
   (c) $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
   (d) if $\mathfrak{p}$ is prime, then $\sqrt{\mathfrak{p}^n} = \mathfrak{p}$ for all $n > 0$.

(2) Let $I, J \lhd R$ be ideals. Then we define the ideal quotient $(I : J) := \{x \in R | xJ \subset I\}$. Prove the following:
   (a) $(I : J)J \subset I$.
   (b) $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$.

(3) Every non-zero homomorphic image of a local ring is local.

(4) Show that $S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$.

(5) Let $\mathfrak{a} \lhd R$ be an ideal and $\mathfrak{p}_i$ be prime ideals. Suppose that $\mathfrak{a} \subset \cup_{i=1}^{n} \mathfrak{p}_i$. Show that $\mathfrak{a} \subset \mathfrak{p}_i$ for some i.

(6) Let R be a non-zero ring. Show that the set of prime ideals of R has minimal elements with respect to inclusion.

(7) Complete the exercises in the note.