Advanced Algebra II

Final Examination

due on Jun. 19, 2004

All rings are commutative with identity unless otherwise stated.

(1) Prove that $\mathbb{C}$ is algebraically closed.

(2) Prove that the only division rings finite dimensional over $\mathbb{R}$ are $\mathbb{H}, \mathbb{C}, \mathbb{R}$.

(3) Let A be a domain. Show that the following are equivalent:

  (a) A is integrally closed.

  (b) $A_{\mathfrak{p}}$ is integrally closed for all prime ideal $\mathfrak{p}$.

  (c) $A_{\mathfrak{m}}$ is integrally closed for all maximal ideal $\mathfrak{m}$

  Now let $A = k[x,y]/(y^2 - x^2(x-1))$, where k is algebraically closed. Find all maximal ideal $\mathfrak{m}$ of A such that $A_{\mathfrak{m}}$ is not integrally closed.

(4) Prove or disprove:

  (a) Let $(A\mathfrak{m}), (B, \mathfrak{n})$ be local rings and $\mathfrak{m}, \mathfrak{n}$ are the maximal ideals respectively. Let $\varphi : A \to B$ be a non-zero homomorphism. Then $\varphi^{-1}\mathfrak{n} = \mathfrak{m}$.

  (b) Let K be a field. For any integer $n > 0$ there is an irreducible polynomial of degree n in $K[x]$.

  (c) Let $\mathfrak{N}$ be the nil radical of R. Then $\mathfrak{N} = \cap_{\mathfrak{p} \in \mathrm{Spec} R} \mathfrak{p}$.

  (d) Let I, J be $\mathfrak{p}$-primary ideals. Then there exist n such that $I^n \subset J$.

(5) Let k be an algebraically closed field. And $A = k[x,y,x]$. We consider an ideal $I := (x^2 - yz, xz - x)$. Find a primary decomposition for I.

(6) Let A be a local ring and M, N are finitely generated A-module. Prove that if $M \otimes N = 0$ then $M = 0$ or $N = 0$.

(7) Let R be a ring (not necessarily commutative) which is a finite dimensional algebra over a field. Show that R is Artinian.