Advanced Algebra II

Jacobson radical and semisimplicity

Let R be a ring, we would like to measure how far it is for R being primitive. We will define the "Jacobson radical" $\mathfrak{J}(R) \lhd R$ which basically achieve our goal.

Let's recall that for a simple module M, $M \cong Rx$ for $x \neq 0$. If $A(M) = 0$, then R is primitive. We thus consider

Definition 0.1. *The Jacobson radical is defined as*

$$\mathfrak{J}(R) := \cap_{M:\text{simple module}} A(M).$$

We can analyze $\mathfrak{J}(R)$ a little bit further. Note that a simple module M can be realized as R/I (as left ${}_R\mathfrak{M}$) where I is a regular maximal left ${}_R\mathfrak{M}$. Thus $A(M) = A(R/I)$. We define $(I : R) := \{r \in R | rR \subset I\}$. Then it's easy to see that $A(R/I) = (I : R)$. Therefore we have

$$\mathfrak{J}(R) = \cap_{I:\text{regular maximal left ideal of} R} (I : R).$$

Exercise 0.2. *If I is a regular left ideal, then $(I : R)$ is the largest two-sided ideal in I.*

We can have a refined characterization of Jacobson radical

Theorem 0.3.

$$\mathfrak{J}(R) = \cap_{I:\text{regular maximal left ideal of} R} I.$$

Proof. Let $J' := \cap_{I:\text{regular maximal left ideal of} R} I$. It's clear that $(I : R) \subset I$ for I regular, hence $\mathfrak{J}(R) \subset J'$. If $x \in J'$, we claim that x is left quasi-regular, i.e. $yx + y + x = 0$ for some y. To see this, we consider $\rho_0 := \{rx + r | r \in R\}$. Clearly, ρ_0 is a left ideal. ρ_0 is regular by taking $e = -x$, then $r - re = r + rx \in \rho_0$ for all $r \in R$. Since ρ_0 is regular, by Zorn's Lemma, one can prove that there is a maximal (proper) left ideal containing ρ_0 if $\rho_0 \neq R$. Suppose that $\rho_0 \neq R$, $x \in J' \subset \rho_1$, $rx \in \rho_1$ thus $r \in \rho_1$ for all $r \in R$. This is a contradiction, hence $\rho_0 = R$. It follows that take $-x \in R$, $-x = yx + y$ for some y and we are done.

Suppose now that $J' \subsetneqq \mathfrak{J}(R)$. Then $J' \not\subseteq A(M)$ for some simple module M. That is, $J'M \neq 0$ for some M. In particular, for some $m \neq 0 \in M$, $J'm = M$. There exist $x \in J'$ such that $xm = -m$. Then

$$0 = (x + y + yx)m = xm + ym + yxm = xm + ym - ym = xm = -m,$$

which is a contradiction. We therefore proved that $J' = \mathfrak{J}(R)$. □

Proposition 0.4. $\mathfrak{J}(R/\mathfrak{J}(R)) = 0$.

Proof. Let $\overline{R} := R/\mathfrak{J}(R)$. Since maximal left ideal of $\overline{R}$ corresponds to maximal left ideal of R containing J. Moreover, if I is a regular left ideal of R, then $\overline{I}$ is a regular ideal. To see this, one notes that if there is $e \in R$ such that $r - re \in I$ for all $r \in R$, then $\bar{r} - \bar{r}\bar{e} \in \overline{I}$. Hence one has

$$\mathfrak{J}(\overline{R}) = \cap_{\rho:\text{regular maximal left ideal of}\overline{R}}\rho$$
$$\subset \cap_{I:\text{regular maximal left ideal of}R}\overline{I} = \overline{\mathfrak{J}(R)} = 0.$$

□

Definition 0.5. *R is said to be semisimple if $\mathfrak{J}(R) = 0$. And R is radical if $\mathfrak{J}(R) = R$.*

We are now going to justify that $\mathfrak{J}(R)$ "measure how far it it for R being primitive. Recall that a primitive ring is a ring admitting a faithful simple module. Indeed, if M is a simple R-module, then M is a faithful simple $R/A(M)$-module. Hence $R/A(M)$ is a primitive ring. So if M is simple, then $A(M)$ has the property that $A(M)$ is primitive. In general, we can define

Definition 0.6. *An ideal $I \lhd R$ is said to be primitive if R/I is a primitive ring.*

Conversely, let $I \lhd R$ be a primitive ideal such that R/I is a primitive ring with faithful simple module M. Then we can define a module structure $R \times M \to M$ naturally via the structure $R/I \times M \to M$. (More precisely, $(r, x) := \bar{r}x$). So M is also a simple R-module. One checks that $A(M) = I$. That is, a primitive ideal must be of the form $A(M)$ for some simple module M.

In particular, we have

$$\mathfrak{J}(R) = \cap_{I:\text{primitive ideal}}I.$$

We can now describe a semisimple as:

Proposition 0.7. *R is semisimple if and only if R is isomorphic to a subdirect product of primitive rings, that is, $R \subset \prod R_i$ with R_i primitive and $\pi_k(R) = R_k$ is surjective for all $\pi_k : \prod R_i \to R_k$.*

Proof. We consider $\varphi : R \to \prod_{I:\text{rm primitive}} R/I$. It's clear that $\pi_k : R \to R/I_k$ is surjective. It's also clear that $\text{Ker}(\varphi) = \mathfrak{J}(R)$. Thus φ is injective if R is semisimple.

Conversely, if R isomorphic to a subdirect product in $\prod R_i$ with R_i primitive. Let $I_k := \text{Ker}(\pi_k)$, then I_k is a primitive ideal. We have

$$\mathfrak{J}(R) = \cap_{I:\text{rm primitive ideal}}I \subset \cap_k I_k = 0.$$

Thus R is semisimple. □

Theorem 0.8 (Wedderburn-Artin Theorem)**.** *R is a semisimple left Artinian ring if and only if there are division rings $D_1, ..., D_r$ and positive integers $n_1, ..., n_r$ such that*

$$R \cong \prod_{i=1}^{r} Mat_{n_i}(D_i).$$

For the proof, we need the following version of

Theorem 0.9 (Chinese Remainder Theorem)**.** *Let* $I_i \lhd R$ *be ideals of* R *such that*

(1) $R^2 + I_i = R$ *for all* i*.*
(2) $I_i + I_j = R$ *for all* $i \neq j$*.*

Then

$$R/(\cap I_i) \cong \prod R/I_i.$$

Proof. Recall that $\mathfrak{J}(R) = \cap P_i = 0$, where P_i are primitive ideals. R/P_i is a Artinian primitive ring, hence a simple ring. Thus $(R/P_i)^2 \neq 0$. Hence $R^2 + P_i = R$. Moreover, R/P_i has no proper ideals, hence P_i is maximal. We therefore have $P_i + P_j = R$ for $i \neq j$. By Chinese Remainder Theorem,

$$R = R/\mathfrak{J}(R) \cong \prod R/P_i.$$

R is Artinian, so it's a finite product. Each R/P_i is an Artinian simple ring, hence isomorphic to $Mar_{n_i}(D_i)$ for some division ring D_i.

Conversely, if $R \cong \prod_{i=1}^{t} Mat_{n_i}(D_i)$, then

$$\mathfrak{J}(R) \cong \prod \mathfrak{J}(Mat_{n_i}(D_i)) = 0.$$

Hence R is semisimple. Each $Mat_{n_i}(D_i)$ is Artinian, so is R. □