Advanced Algebra II

STRUCTURE OF RINGS

Let M be a simple ${}_R\mathfrak{M}$, by Schur's Lemma, $\text{End}_R(M) = \text{Hom}_R(M, M)$ is a division ring, call it D. Then M is naturally a D-module as we have

$$D = \text{End}_R(M) \times M \to M.$$

With this, we are able to analyze the structure of primitive rings. Recall that R is primitive if there exist a faithful simple module M. Since M is faithful, one has an embedding $\Phi : R \hookrightarrow \text{End}(M)$ by $\Phi(r) = T_r$, where $T_r(x) := rx$. For $\psi \in D = C(M) = \text{End}_R(M)$, we have

$$\psi T_r(x) = T_r\psi(x) \forall x \in M.$$

Therefore, T_r is D-linear, i.e. $\Phi : R \hookrightarrow \text{End}_D(M)$. If M is a finite dimensional vector space over D of dimension n, then $\text{End}_D(M)$ is nothing but $Mat_n(D)$.

A natural question is that whether a primitive ring $R = \text{End}_D(M)$? The answer is YES under some finiteness condition. This is the content of Wedderburn-Artin Theorem. However, this is not the case in general. Take the example as in Example 0.11, I is a ring and V is faithful simple I-module. Thus I is primitive but not equal $\text{End}_D(M)$. Therefore, we need the refined notion of *density* to characterize primitive ring.

Definition 0.1. *Let V be a vector space over a division ring D. A subring $R \subset \text{End}_D(V)$ is called a dense subring if for every positive integer n, every linearly independent subset $\{u_1, ..., u_n\}$ of V and every arbitrary subset $\{v_1, ..., v_n\}$ of V, there is $\theta \in R$ such that $\theta(u_i) = v_i$ for all $i = 1, ..., n$.*

Theorem 0.2 (Jacobson Density Theorem)**.** *Let R be a primitive ring with a faithful simple module M. And let D be the division ring $\text{End}_R(M)$. Then R is isomorphic to a dense subring of $\text{End}_D(M)$. Conversely, let M be a vector space over a division ring D, then a dense subring of $\text{End}_D(M)$ is primitive.*

Proof. We have seen that $\Phi : R \hookrightarrow \text{End}_D(M)$. It suffices to show that $\Phi(R)$ is dense.
Claim. Let V be a finite dimensional subspace of M, then for $x \in MV$, there exist $r \in R$ such that $rx \neq 0$ and $rV = 0$.

Grant this for the time being. For a linearly independent subset $\{u_1, ..., u_n\}$. Let V be the vector space spanned by it, and V_i be the space spanned by $\{u_1, ., u_{i-1}, u_{i+1}, .., u_n\}$. By the claim, there is r_i such that $r_iu_i \neq 0$, $r_iu_j = 0$ for all $j \neq i$. Consider now $Rr_iu_i < M$. Apply the claim to $V = 0$, then $Rr_iu_i \neq 0$. By the simplicity, $Rr_iu_i = M$. In particular, for $v_i \in M$, there is $s_i \in R$ such that $s_ir_iu_i = v_i$. We now have $\theta := \sum s_ir_i$ satisfying the required property.

The converse statement is easy.

We now prove the claim by induction on $\dim_D V$. First of all, it's clearly true if $\dim V = 0$. Suppose that $V = V' + Dw$ for some $w \notin V'$ and the claim is true for V'. Let $A(V')$ be the annihilator of V' in R, which is a left ideal of R. The induction hypothesis asserts that there is $r \in A(V')$ and $rw \neq 0$. In particular, $A(V')w < M$ is a non-zero submodule. By the simplicity of M, $A(V')w = M$.

Let $S := \{m \in M, m \notin V | A(V)m = 0\}$. We need to show that $S = \emptyset$. Say if $m \in S$, we can define $\tau : M \to M$ by: if $x \in M$ hence $x = aw$ for some $a \in A(V')$. We define $\tau(x) := am$. Note that if $x = a_1 w = a_2 w$, then $a_1 - a_2 \in A(w) \cap A(V') = A(V)$. One has $a_1 m - a_2 m \in A(V)m = 0$. Thus τ is well-defined.

If $x = aw$ with $a \in A(V')$, then $rx = raw$. So $\tau(rx) = ram = r(am) = r\tau(x)$. So $\tau \in D$. Therefore, for all $a \in A(V')$, $a(m - \tau(w)) = 0$. So $m - \tau(w) \in V'$, otherwise, by induction hypothesis, there exist $b \in A(V')$ which does not annihilates $m - \tau(w)$. Thus, $m \in \tau(w) + V' \subset Dw + V' = V$. This is the required contradiction. $\square$

Definition 0.3. *A (left) module $M \in {}_R\mathfrak{M}$ is said to be Artinian (resp. Noetherian) if it satisfies descending chain condition (resp. ascending) on submodules.*

A ring R is said to be left Artinian (resp. Noetherian) if R is Artinian (resp. Noetherian) as left R-module.

R is said to be Artinian (resp. Noetherian) if R is both left and right Artinian (resp. Noetherian).

Theorem 0.4. *The following condition on left Artinian rings are equivalent:*

1. *R is simple.*
2. *R is primitive.*
3. *$R \cong \mathrm{End}_D(V)$ for a non-zero finite dimensional vector space over D.*
4. *$R \cong Mat_n(D)$ for some positive integer n and division ring D.*

Proof. (1) $\Rightarrow$ (2) Let $I := \{r \in R | Rr = 0\}$ the right annihilator of R. It's clear that $I \lhd R$. R is simple, thus $I = 0$. R is left Artinian, thus there is a minimal left non-zero ideal, say J. Now $A(J) \lhd R$. If $A(J) = R$, then $J \subset I = 0$ which is absurd. Thus $A(J) = 0$. So J is a faithful simple ${}_R\mathfrak{M}$, and hence R is primitive.

(2) $\Rightarrow$ (3) By density theorem, R is isomorphic to a dense subring of $T \subset \mathrm{End}_D(V)$. We claim that $\dim_D(V)$ is finite. Suppose not, then there is an infinite sequence of independent subset $\{u_1, u_2, ...\}$. Let I_n be the left annihilator of $\{u_1, ..., u_n\}$. One sees that $I_1 \supsetneq I_2 \supsetneq ...$ give a proper descending chain, which is the required contradiction. One next show that a dense subring of $\mathrm{End}_D(V)$ is the whole ring if $\dim_D(V)$ is finite.

(3) $\Leftrightarrow$ (4) trivial. And we have seen (4) $\Rightarrow$ (1) before.

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