

Advanced Algebra II

LOCALIZATION

Before we going to the study of dimension theory, we need to recall some basic notion of localization.

Definition 0.1. A subset $S \subset R$ is said to be a multiplicative set if

- (1) $1 \in S$,
- (2) if $a, b \in S$, then $ab \in S$.

Given a multiplicative set, then one can construct a localized ring $S^{-1}R$ which I suppose the readers have known this. In order to be self-contained, I recall the construction:

In $R \times S$, we define an equivalent relation that $(r, s) \sim (r', s')$ if $(rs' - r's)t = 0$ for some $t \in S$. Let $\frac{r}{s}$ denote the equivalent class of (r, s) . One can define addition and multiplication naturally. The set of all equivalent classes, denoted $S^{-1}R$, is thus a ring. There is a natural ring homomorphism $\iota : R \rightarrow S^{-1}R$ by $\iota(r) = \frac{r}{1}$.

Remark 0.2.

- (1) If $0 \in S$, then $S^{-1}R = 0$. We thus assume that $0 \notin S$.
- (2) If R is a domain, then ι is injective. And in fact, $S^{-1}R \hookrightarrow F$ naturally, where F is the quotient field of R .
- (3) Let $J \triangleleft S^{-1}R$. We will use $J \cap R$ to denote the ideal $\iota^{-1}(J)$. (If R is a domain, then $J \cap R = \iota^{-1}(J)$ by identifying R as a subring of $S^{-1}R$).

I would like to recall the most important example and explain their geometrical meaning, which, I think, justify the notion of localization.

Example 0.3. Let $f \neq 0 \in k[x_1, \dots, x_n]$ and let $S = \{1, f, f^2, \dots\}$. The localization $S^{-1}k[x_1, \dots, x_n]$ is usually denoted $k[x_1, \dots, x_n]_f$. This ring can be regarded as "regular functions" on the open set $U_f := \mathbb{A}_k^n - \mathcal{V}(f)$. One notices that U_f is of course the maximal open subset that the ring $k[x_1, \dots, x_n]_f$ gives well-defined functions.

Example 0.4. Let $x = (a_1, \dots, a_n) \in \mathbb{A}_k^n$ and $\mathfrak{m}_x = (x_1 - a_1, \dots, x_n - a_n)$ be its maximal ideal. Take $S = k[x_1, \dots, x_n] - \mathfrak{m}_x$, then the localization is denoted $k[x_1, \dots, x_n]_{\mathfrak{m}_x}$. It is the ring of regular functions "near x ".

Recall that for a R -module M , one can also define $S^{-1}M$ which is naturally an $S^{-1}R$ -module. And we have:

Proposition 0.5.

- (1) If $I \triangleleft R$, then $S^{-1}I \triangleleft S^{-1}R$. Moreover, every ideal $J \triangleleft S^{-1}R$ is of the form $S^{-1}I$ for some $I \triangleleft R$.
- (2) For $J \triangleleft S^{-1}R$, then $S^{-1}(J \cap R) = J$.
- (3) $S^{-1}I = S^{-1}R$ if and only if $I \cap S \neq \emptyset$.

- (4) *There is a one-to-one correspondence between $\{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \cap S = \emptyset\}$ and $\{\mathfrak{q} \in \text{Spec}(S^{-1}R)\}$.*
- (5) *In particular, the prime ideals of the local ring $R_{\mathfrak{p}}$ are in one-to-one correspondence with the prime ideals of R contained in \mathfrak{p} .*

Proof.

- (1) If $\frac{x}{s}, \frac{y}{t} \in S^{-1}I$, that is, $x, y \in I$, then $\frac{x}{s} + \frac{y}{t} = \frac{xt+ys}{st} \in S^{-1}I$. And if $\frac{r}{s} \in S^{-1}R$ and $\frac{x}{t} \in S^{-1}I$, then $\frac{r}{s} \frac{x}{t} = \frac{rx}{st} \in S^{-1}I$. Hence $S^{-1}I$ is an ideal.

Moreover, let $J \triangleleft S^{-1}R$. Let $I := \iota^{-1}(J) \triangleleft R$. We claim that $J = S^{-1}I$. To see this, for $x \in I$, $\frac{x}{1} \in J$. Hence $\frac{x}{s} = \frac{x}{1} \frac{1}{s} \in J$ for all $s \in S$. It follows that $S^{-1}I \subset J$. Conversely, if $\frac{x}{s} \in J$, then $\frac{x}{1} = \frac{x}{s} \frac{s}{1} \in J$ and $\frac{x}{1} = \iota(x)$. So $\frac{x}{s} \in S^{-1}I$.

- (2) If $\frac{x}{s} \in J \cap R$, then $\frac{x}{s} = \frac{y}{1}$ for some $y \in R$, i.e. $(x - sy)t_0 = 0$, for some $t_0 \in S$. Then look at $\frac{y}{t} \in S^{-1}(J \cap R)$. It's clear that $\frac{y}{t} = \frac{x}{s} \frac{1}{t} \in J$.

Conversely, if $\frac{x}{s} \in J$, then $\frac{x}{1} \in J \cap R$. And hence $\frac{x}{s} \in S^{-1}(J \cap S)$.

- (3) If $x \in I \cap S$, then $\frac{1}{1} = \frac{x}{x} \in S^{-1}I$. Conversely, if $\frac{1}{1} = \frac{x}{s} \in S^{-1}I$, then $(x - s)t = 0$. Therefore, $xt = st \in S \cap I$.
- (4) For $\mathfrak{q} \in \text{Spec}(S^{-1}R)$. It's clear that $\mathfrak{q} \cap R = \iota^{-1}\mathfrak{q} \in \text{Spec}(R)$. $(\mathfrak{q} \cap R) \cap S = \emptyset$, otherwise $\mathfrak{q} = S^{-1}(\mathfrak{q} \cap R) = S^{-1}R$ which is impossible.

Conversely, let $\mathfrak{p} \in \text{Spec}(R)$ and $\mathfrak{p} \cap S = \emptyset$. We would like to show that $S^{-1}\mathfrak{p}$ is a prime ideal. First of all, if $\frac{1}{1} \in S^{-1}\mathfrak{p}$, then $\frac{1}{1} = \frac{x}{s}$ for some $x \in \mathfrak{p}$. It follows that $(x - s)t = 0$ for some $t \in S$. Thus $st = xt \in \mathfrak{p}$ which is a contradiction. Therefore, $S^{-1}\mathfrak{p} \neq S^{-1}R$.

Moreover, if $\frac{xy}{st} \in S^{-1}\mathfrak{p}$. Then $\frac{xy}{st} = \frac{x'}{s'}$ for some $x' \in \mathfrak{p}$ and $s' \in S$. Then $(xys' - stx')t' = 0$ for some $t' \in S$. Hence $xys't' \in \mathfrak{p}$. It follows that $xy \in \mathfrak{p}$ since $s't' \notin \mathfrak{p}$. Thus either x or y in \mathfrak{p} , and either $\frac{x}{s}$ or $\frac{y}{t}$ in $S^{-1}\mathfrak{p}$.

It remains to show that the correspondence is a one-to-one correspondence. We have seen that

$$\iota^{-1} : \text{Spec}(S^{-1}R) \rightarrow \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \cap S = \emptyset\}$$

is surjective. By (2), it follows that this is injective.

- (5) Let $S = R - \mathfrak{p}$, then $S \cap \mathfrak{q} = \emptyset$ if and only if $\mathfrak{q} \subset \mathfrak{p}$.

□

But for $I \triangleleft R$, then $S^{-1}I \cap R \supset I$ only. Indeed, if $x \in I \triangleleft R$. Then $x = \frac{xs}{x} \in S^{-1}I \cap R$. Conversely, for $x \in S^{-1}I \cap R$, then $\frac{x}{1} = \frac{y}{t}$ for some $y \in I$. Thus $(y - xt)s = 0$ for some $s, t \in S$. We can not get $y \in I$ in general. However, this is the case if I is prime and $S \cap I = \emptyset$. Thus we have

Proposition 0.6. *If $\mathfrak{p} \triangleleft R$ is a prime ideal and $S \cap \mathfrak{p} = \emptyset$. Then $S^{-1}\mathfrak{p} \cap R = \mathfrak{p}$.*

Corollary 0.7. *If R is Noetherian (resp. Artinian), then so is $S^{-1}R$.*

Proof. This is an immediate consequence of (1) of the Proposition. \square

Example 0.8. *Let $\mathfrak{p} \in \text{Spec}(R)$, then $R_{\mathfrak{p}}$ is a local ring with the unique maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. To see that, if there is a maximal ideal \mathfrak{m} . By the correspondence, $\mathfrak{m} = \mathfrak{q}R_{\mathfrak{p}}$ for some $\mathfrak{q} \subset \mathfrak{p}$. Thus $\mathfrak{m} \subset \mathfrak{p}R_{\mathfrak{p}}$ and thus must be equal.*

A ring with a unique maximal ideal is called a local ring. Thus $R_{\mathfrak{p}}$ is a local ring.

Proposition 0.9. *The operation S^{-1} commutes with formation of finite sums, product, intersection and radicals.*

Proof. These can be checked directly. \square

Another important feature is that

Proposition 0.10. *The operation S^{-1} is exact. That is, if*

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

is an exact sequence of R -module, then

$$S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$$

is exact as $S^{-1}R$ -module.

Proof. By the construction, it follows directly that $S^{-1}g \circ S^{-1}f = 0$. It suffices to check that $\text{Ker}(S^{-1}g) \subset \text{Im}(S^{-1}f)$. If $\frac{x}{s} \in \text{Ker}(S^{-1}g)$, then $\frac{g(x)}{s} = 0 \in S^{-1}M''$. That is $tg(x) = 0$ for some $t \in S$. We have $tg(x) = g(tx) = 0$. And then $tx = f(y)$ for some $y \in M'$. Therefore,

$$\frac{x}{s} = \frac{xt}{st} = \frac{f(y)}{st} = S^{-1}f\left(\frac{y}{st}\right).$$

\square

Corollary 0.11. *The operation S^{-1} commutes with passing to quotients by ideals. That is, let $I \triangleleft R$ be an ideal and \bar{S} the image of S in $\bar{R} := R/I$. Then $S^{-1}R/S^{-1}I \cong \bar{S}^{-1}\bar{R}$.*

Proof. By considering $0 \rightarrow I \rightarrow R \rightarrow \bar{R} \rightarrow 0$ as an exact sequence of R -modules. We have $S^{-1}R/S^{-1}I \cong S^{-1}\bar{R}$ as $S^{-1}R$ -modules. We claim that there is a natural bijection from $S^{-1}\bar{R}$ to $\bar{S}^{-1}\bar{R}$ by $\frac{\bar{x}}{s} \mapsto \frac{\bar{x}}{\bar{s}}$ which is compatible with all structures.

One can also try to prove this directly. Basically, construct a surjective ring homomorphism $S^{-1}R \rightarrow \bar{S}^{-1}\bar{R}$ and shows that the kernel is $S^{-1}I$. We leave it as an exercise. \square