

Advanced Algebra II

NOETHERIAN RINGS AND PRIMARY DECOMPOSITION

We will assume R is commutative with identity in the following. One can define the similar notion of Noetherian rings and Artinian rings as:

Definition 0.1. *A ring R is said to be Noetherian (resp. Artinian) if R is a Noetherian (resp. Artinian) R -module.*

Even though the definition looks quite similar to modules. We would like to remark that there is some subtlety here.

Example 0.2. *We know that a submodule of a Noetherian module is Noetherian. But a subring $S \subset R$ of a Noetherian ring R is not necessarily a Noetherian ring. For example, let $S = k[x_1, \dots]$ be a polynomial ring with infinitely many indeterminates and R be its quotient field. R is a Noetherian ring but S is not.*

We will first examine some properties of Noetherian ring. It might be surprising at the first glance that an Artinian ring is Noetherian.

Proposition 0.3. *If R is Noetherian ring and $I \triangleleft R$ is an ideal, then R/I is Noetherian.*

Proof. Since ideals of R/I is of the form J/I for some $J \triangleleft R$. An ascending chain of ideal in R/I corresponds to an ascending chain of ideals in R hence stationary. \square

Theorem 0.4 (Hilbert Basis Theorem). *If R is Noetherian, then the polynomial ring $R[x]$ is Noetherian.*

Proof. Let $I \triangleleft R$ be an ideal. We define

$$I_0 := \{a \in R \mid f = ax^n + l.o.t \in I\}.$$

It's clear that $I_0 \triangleleft R$. Let b_1, \dots, b_r be a set of generator of I_0 . And let $f_i = b_i x^{n_i} + l.o.t$ be a polynomial with leading coefficient b_i . Let $n = \max\{n_i\}$. If $f \in I$ has degree $\geq n$ with leading coefficient b , then $b = \sum s_i b_i$ for some $s_i \in R$. It follows that we can "divide" f by f_i

$$f = \sum s_i x^{n-n_i} f_i + g.$$

Where $g \in I$ and $\deg(g) < \deg(f)$. Inductively, one can write $f = \sum g_i f_i + h$ with $\deg(h) < n$.

Let $M = R + Rx + \dots + Rx^{n-1}$ which is a Noetherian R -module. Then $I \cap M$ is a Noetherian R -module. Let h_1, \dots, h_t be a set of generators of $I \cap M$ as R -module. Then we have seen that I is generated by $f_1, \dots, f_r, h_1, \dots, h_t$ as $R[x]$ -module. Thus $R[x]$ is Noetherian. \square

Corollary 0.5. *Let R be a Noetherian ring. Then any finitely generated R -algebra is Noetherian.*

Proof. If S is a finitely generated R -algebra, say generated by s_1, \dots, s_n , then there is a surjective ring homomorphism

$$\begin{aligned}\varphi : R[x_1, \dots, x_n] &\rightarrow S = R[s_1, \dots, s_n] \\ f(x_1, \dots, x_n) &\mapsto f(s_1, \dots, s_n).\end{aligned}$$

By Hilbert Basis Theorem, $R[x_1, \dots, x_n]$ is a Noetherian ring. And hence the homomorphic image $S \cong R[x_1, \dots, x_n]/\text{Ker}\varphi$ is a Noetherian ring. \square

Example 0.6. Let k be a field. We consider $\mathbb{A}_k^n := \{(a_1, \dots, a_n) \mid a_i \in k\}$ the affine space. For a subset $S \subset \mathbb{A}^n$, one can consider

$$\mathcal{I}(S) := \{f \in k[x_1, \dots, x_n] \mid f(x) = 0, \forall x \in S\}.$$

It's clear that $\mathcal{I}(S) \triangleleft k[x_1, \dots, x_n]$ and hence finitely generated. The ring $A(S) := k[x_1, \dots, x_n]/\mathcal{I}(S)$ represents "polynomial functions on S " is a Noetherian ring.

By the similar argument as in the proof of Hilbert Basis Theorem, one can show that: (while leading term now is the term of smallest degree)

Proposition 0.7. Let R be a Noetherian ring. Then $R[[x]]$ is a Noetherian ring.

An important consequence is the following:

Corollary 0.8. Let R be an Noetherian ring and S be a finitely generated R -algebra, then S is finitely generated.

Another important feature of Noetherian ring is that it allows us to perform the *primary decomposition*. As we'll see later in this section that there is a nice correspondence between *algebra* (polynomial rings) and *geometry* (algebraic sets). The primary decomposition in the algebraic side coincide with the decomposition of an algebraic set into union of *irreducible components*.

We first work on the primary decomposition for Noetherian rings.

Definition 0.9. An ideal $\mathfrak{q} \triangleleft R$ is said to be *primary* if $xy \in \mathfrak{q}$ then either $x \in \mathfrak{q}$ or $y^n \in \mathfrak{q}$ for some n .

Proposition 0.10. If \mathfrak{q} is primary, then $\sqrt{\mathfrak{q}}$ is a prime ideal. Moreover, every prime ideal containing \mathfrak{q} contains $\sqrt{\mathfrak{q}}$.

Let $\mathfrak{p} := \sqrt{\mathfrak{q}}$, then we say \mathfrak{q} is \mathfrak{p} -primary.

Proof. Let $\mathfrak{p} := \sqrt{\mathfrak{q}}$. We first show that \mathfrak{p} is prime. To this end, if $xy \in \mathfrak{p}$, then $x^n y^n = (xy)^n \in \mathfrak{q}$ for some n . Since \mathfrak{q} is primary, one has either $x^n \in \mathfrak{q}$ or $(y^n)^m \in \mathfrak{q}$ for some m . In any case, one has either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

Let \mathfrak{p}' be a prime ideal containing \mathfrak{q} . If $x \in \mathfrak{p} = \sqrt{\mathfrak{q}}$, then $x^n \in \mathfrak{q} \subset \mathfrak{p}'$ for some n . It follows that $x \in \mathfrak{p}'$. Hence we have $\mathfrak{p} \subset \mathfrak{p}'$. \square

Definition 0.11. Let $\mathfrak{a} \triangleleft R$ be an ideal. We say that \mathfrak{a} is irreducible if there is no non-trivial decomposition, i.e. for any $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$, then $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{c}$.

Proposition 0.12. If R is Noetherian, then every ideal can be written as intersection of finitely many irreducible ideals.

Proof. Let $\mathfrak{a} \triangleleft R$ be an ideal. Suppose that \mathfrak{a} is irreducible, then nothing to prove. If \mathfrak{a} is not irreducible, then $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}'$. If both \mathfrak{a}_1 and \mathfrak{a}' are irreducible, then we are done. Otherwise, we may assume that \mathfrak{a}_1 is reducible. Note that $\mathfrak{a} \subset \mathfrak{a}_1$. By continuing this process, we get a sequence of ideals

$$\mathfrak{a} \subset \mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \dots$$

Since R is Noetherian, this process must terminate and hence we are done.

Another way to put it is: Let Σ be the set of ideals which cannot be written as decomposition of irreducible ideals. We would like to prove that Σ is empty. If $\Sigma \neq \emptyset$, by the maximal condition, there is a maximal element $\mathfrak{a} \in \Sigma$. $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ since \mathfrak{a} is not irreducible. By the maximality of \mathfrak{a} , one has that both \mathfrak{b} and \mathfrak{c} have finite decomposition, hence so is \mathfrak{a} . This is the required contradiction. \square

Proposition 0.13. If R is a Noetherian ring, then irreducible ideal is primary.

Proof. Let $\mathfrak{a} \triangleleft R$ be an irreducible ideal. We need to show that \mathfrak{a} is primary. Let's pass to the ring $\bar{R} := R/\mathfrak{a}$. It suffices to show that if $xy = 0 \in \bar{R}$, then either $x = 0$ or $y^n = 0$ for some n .

Claim. $0 = (x) \cap (y^n)$.

Grant this claim, then by the irreducibility of \mathfrak{a} , one has that $0 \triangleleft \bar{R}$ is irreducible. Hence $0 = (x)$ or $0 = (y^n)$ and we are done.

To prove the claim, we consider the ascending chain

$$\text{Ann}(y) \subset \text{Ann}(y^2) \subset \dots$$

By the a.c.c., there is n such that $\text{Ann}(y^n) = \text{Ann}(y^{n+1})$. If $a \in (x) \cap (y^n)$, then $a = bx = cy^n$ for some b, c . $ay = 0$ since $(bx)y = b(xy) = 0$. Hence $cy^n y = cy^{n+1} = 0$. So $c \in \text{Ann}(y^{n+1}) = \text{Ann}(y^n)$. Therefore, $a = cy^n = 0$. \square

Combining all these, we have

Proposition 0.14. Let R be a Noetherian ring, then every ideal in R has a primary decomposition, i.e. can be written as a finite union of primary ideals.

Exercise 0.15. Let $I \triangleleft R$ be a radical ideal in a Noetherian ring R , then $I = \bigcap_{i=1}^r \mathfrak{p}_i$ for some prime ideal \mathfrak{p}_i .