Advanced Algebra II

Tensor product

In this section, we are going to define an important notion, called *tensor product*.

Let's start by recalling that, in linear algebra, there are various situation we need to consider *bilinear maps*. More precisely, let V, W be vector spaces over a field k. We know that all linear maps from V to k again form a vector space which is the dual vector space of V. Moreover, we can consider maps $f : V \times W \to k$ which is linear with respect to V and W. Such maps are called bilinear maps. (One can even consider multilinear maps). We can characterize them in terms of basis.

Now in general, we would like to work on modules over rings. Let $M = M_R$ be a right R-module and $N = {}_RN$ be a left R-module. And let C be an abelian group. We would like to consider maps $f : M \times N \to C$ satisfying

$$\begin{cases} f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b) \\ f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2) \\ f(ar, b) = f(a, rb) \end{cases} \tag{1}$$

for all $a, a_1, a_2 \in M, b, b_1, b_2 \in N, r \in R$. A map satisfying equation (1) is called *middle linear*.

Theorem 0.1 (Existence and universal property of tensor product)**.** *Let R be a ring and $M = M_R$ be a right R-module and $N = {}_RN$ be a left R-module. There is an abelian group, denoted $M \otimes_R N$ together with a middle linear map $\imath : M \times N \to M \otimes_R N$. Moreover, for any middle linear map $f : M \times N \to C$ to an abelian group C, there is a unique homomorphism $\bar{f} : M \otimes_R N \to C$ such that $\bar{f}\imath = f$.*

As an easy exercise of universal property, one can easily see that:

Corollary 0.2. *The tensor product is unique up to isomorphism.*

Proof. We first construct $M \otimes_R N$. Let F be the free abelian group on the set $M \times N$. Let K be the subgroup generated by following elements (for all $a, a_1, a_2 \in M, b, b_1, b_2 \in N, r \in R$):

$$\begin{cases} (a_1 + a_2, b) - (a_1, b) - (a_2, b); \\ (a, b_1 + b_2) - (a, b_1) - (a, b_2); \\ (ar, b) - (a, rb). \end{cases}$$

Let $M \otimes_R N := F/K$. Clearly, elements in $M \otimes_R N$ is generated by cosets $(a, b) + K$. We denote the coset $(a, b) + K$ by $a \otimes b$.

Let $\imath : M \times N \to M \otimes_R N$ by $\imath(a, b) = a \otimes b$. It's clear that $\imath$ is middle linear.

It remains to show that $M\otimes_R N$ has the required universal property. To see this, let $f : M \times N \to C$ be a middle linear map. Naturally, we consider $\bar{f} : M\otimes_R N \to C$ by $\bar{f}(a\otimes b) := f(a, b)$. (Or one can say that f induces a homomorphism $f' : F \to C$. K is in the kernel, hence it induces $\bar{f} : F/K \to C$.) The uniqueness actually follows from the universal property of free abelian group F. $\square$

Example 0.3.
$\mathbb{Z}_2\otimes_{\mathbb{Z}}\mathbb{Z}_3 = 0.$
$\mathbb{Z}_2\otimes_{\mathbb{Z}}\mathbb{Z}_2 \cong \mathbb{Z}_2.$

The notion of tensor product is quite general. The point is that it makes sense in various setting.

Example 0.4. *Let R, S be rings. Let $M = M_R$ be a right R-module and $N = {}_RN$ be a left R-module. If $M = {}_SM$ is a left S-module, then $M\otimes_R N$ is a left S-module naturally. Similarly, if $N = N_S$, then $M\otimes_R N$ is a right S-module naturally.*

Proposition 0.5. *Let M_R, M'_R be right R-modules and ${}_RN, {}_RN'$ be left R-modules. And let $f : M \to M'$, $g : N \to N'$ be module homomorphisms. Then there is a unique group homomorphism $f\otimes g : M\otimes_R N \to M'\otimes_R N'$.*

Proof. Consider a middle linear map $\overline{(f, g)} : M \times N \to M'\otimes_R N$ by $(a, b) \mapsto f(a)\otimes g(b)$. By the universal property, we are done. $\square$

Proposition 0.6. *Let R be a ring with identity, then there are R-module isomorphism $M\otimes_R R \cong M$ and $R\otimes_R N \cong N$.*

Proof. There is a natural map $\jmath : N \to R\otimes_R N$ by $\jmath(x) = x\otimes 1$. It's clear that this is an ${}_R\mathfrak{M}$ homomorphism. We then construct $f : R\times N \to N$ by $f(r, x) = rx$. It's clear that this is middle linear and thus induces a group homomorphism $\bar{f} : R\otimes_R N \to N$ by $\bar{f}(r\otimes x) = rx$. It's also easy to see that this is a module homomorphism.

Therefore, it suffices to check that $\bar{f}\jmath = \mathbf{1}_N$ (which is clear) and $\jmath\bar{f} = \mathbf{1}_{R\otimes_R N}$. This mainly due to

$$\sum r_i\otimes x_i = \sum(1\otimes r_i x_i) = 1\otimes \sum r_i x_i.$$

$\square$

Theorem 0.7 (Tensor product is right exact)**.** *Let $N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3 \to 0$ be an exact sequence of left R-module. Then for any right R-module M, we have*

$$M\otimes_R N_1 \xrightarrow{\mathbf{1}\otimes f} M\otimes_R N_2 \xrightarrow{\mathbf{1}\otimes g} M\otimes_R N_3 \to 0$$

is exact.

Proof. For $y \in N_3$, $y = g(z)$ for some $z \in N_2$, thus for $x \in M$, $x \otimes y = (\mathbf{1}\otimes g)(x \otimes z)$. Hence $\mathrm{Im}(\mathbf{1}\otimes g)$ generate $M \otimes_R N_3$. It follows that $\mathbf{1}\otimes g$ is surjective.

$$(\mathbf{1}\otimes g)(\mathbf{1}\otimes f)(x\otimes w) = x \otimes gf(w) = x\otimes 0 = 0.$$

Therefore, $\mathrm{Im}(\mathbf{1}\otimes f) \subset \mathrm{Ker}(\mathbf{1}\otimes g)$. There is thus an induced map $\alpha : M\otimes_R N_2/\mathrm{Im}(\mathbf{1}\otimes f) \to M \otimes_R N_3$. It suffices to show that α is an isomorphism. To this end, we intend to construct the inverse map. Consider $x\otimes y \in M\otimes_R N_3$, there is $z \in N_2$ such that $g(z) = y$. We define $\beta_0 : M \times N_3 \to M\otimes_R N_2/\mathrm{Im}(\mathbf{1}\otimes f)$ by $\beta_0(x, y) = \overline{x\otimes z}$. We first check that this is well-defined. If $z, z' \in N_2$ such that $g(z) = g(z') = y$, then $z - z' \in \mathrm{Ker} g = \mathrm{Im} f$. Thus there is $w \in N_1$ such that $z - z' = f(w)$. One verifies that

$$\overline{x\otimes z} = \overline{x\otimes(z' + f(w))} = \overline{x\otimes z'} + \overline{x\otimes f(w)}$$

$$= \overline{x\otimes z'} + \overline{(\mathbf{1}\otimes f)(x\otimes w)} = \overline{x\otimes z'}.$$

It's routine to check that β_0 is middle linear, hence it induces $\beta : M\otimes_R N_3 \to M\otimes_R N_2/\mathrm{Im}(\mathbf{1}\otimes f)$. One can check that

$$\alpha\beta(x\otimes y) = \alpha\overline{x\otimes z} = x\otimes g(z) = x\otimes y.,$$

$$\beta\alpha(\overline{x\otimes z}) = \beta(x\otimes g(z)) = \overline{x\otimes z}.$$

□

We now concentrate on the case that R is commutative. Let M, N, C be R-modules. By a bilinear map we mean $f : M \times N \to C$ such that

$$\left\{ \begin{array}{l} f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b) \\ f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2) \\ f(ar, b) = f(a, rb) = rf(a, b) \end{array} \right. \tag{2}$$

for all $a, a_1, a_2 \in M, b, b_1, b_2 \in N, r \in R$. One can have $M\otimes_R N$ as in the previous construction. Note that $M\otimes_R N$ is naturally an R-module. And the unique homomorphism $M\otimes_R N \to C$ is an R-module homomorphism.

Example 0.8. *Let V, W be a vector space over k with basis $\{v_1, ..., v_n\}$, $\{w_1, ..., w_m\}$ respectively. Then $V\otimes W$ is a vector space over k with basis $\{v_i\otimes w_j\}_{i=1,...,n, j=1,...,m}$.*

Exercise 0.9 (base change)**.** *Let $\varphi : R \to S$ be a homomorphism of commutative ring with identity. Then S is an R-module. We have $R[x]\otimes_R S \cong S[x]$.*

Theorem 0.10. *Let R, S be rings and $M_R, {}_RN_S, {}_SL$ be modules. Then there is an isomorphism of abelian groups*

$$\mathrm{Hom}_S(M\otimes_R N, L) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(N, L)).$$

We are not going to prove this. The moral for this is that the "functor" tensor product and the "functor" Hom are adjoint. We did see this before in linear algebra:

Example 0.11. *Let V be a vector space over k. Then* $\mathrm{Hom}_k(V \otimes V, k)$ *represents the bilinear maps to k. It's adjoint,* $\mathrm{Hom}_k(V, \mathrm{Hom}_k(V, k)) = \mathrm{Hom}_k(V, V^*)$*, represents the linear maps from V to V^*, where V^* denote the dual vector space of V.*