Advanced Algebra II

Homework 7

due on Apr. 30, 2004

(1) Let $I \subset R$ be a left ideal and let $(I : R) := \{x \in R | xR \subset I\}$.
  (a) Show that $(I : R) \lhd R$
  (b) If I is regular, show that $(I : R)$ is the largest 2-sided ideal of R in I.

(2) Let M be a semisimple ${}_R\mathfrak{M}$ such that $M \cong \oplus_{i=1}^{n} M_i$ with M_i simple. Prove that the decomposition is unique up to isomorphism and reindexing.

(3) Let K_1, K_2 be fields. Show that if $Mat_{n_1}(K_1) \cong Mat_{n_2}(K_2)$ then $K_1 \cong K_2$ and $n_1 = n_2$. (Hint: Schur's Lemma)

(4) Let G be a finite group and K is a field. We consider the group algebra $R := K[G]$ in this problem.
  (a) Show that every left ideal of $K[G]$ is a vector space over K. (Thus left ideals are left "algebra ideals").
  (b) Show that $K[G]$ is left Artinian.
  (c) Let V be a ${}_R\mathfrak{M}$. And $W \subset V$ be a submodule. Show that W is a vector subspace.
  (d) Let $\pi : V \to W$ be a projection of vector spaces. Note that π is not necessarily a ${}_R\mathfrak{M}$-homomorphism. Let $Tr_G(\pi) := \sum_{\sigma \in G} \sigma\pi$. Show that $Tr_G(\pi) : V \to W$ is an ${}_R\mathfrak{M}$-homomorphism.
  (e) Assume that $char(K) \nmid |G|$. Show that the homomorphism $\frac{1}{|G|} Tr_G(\pi) : V \to W$ splits the sequence $0 \to W \to V$.
  With all these, one sees that every ${}_R\mathfrak{M}$ is semisimple. Thus the group algebra $K[G]$ is semisimple. This is the Maschke's Theorem.
  (f) In this case, show that $K[G] \cong \prod_{i=1}^{r} Mat_{n_i}(K)$ with $|G| = \sum_{i=1}^{r} n_i^2$.