Advanced Algebra II

Homework 11

due on May. 28, 2004

(1) Let $S \subset R$ be a multiplicative set and $I \lhd R$ an ideal. Let $\bar{R} := R/I$ and $\bar{S}$ be the image of S in $\bar{R}$.
   (a) Show that $\bar{S}$ is a multiplicative set.
   (b) Show that there is a natural surjective homomorphism $S^{-1}R \to \bar{S}^{-1}\bar{R}$ with kernel $S^{-1}I$.

(2) Let F be a quadratic field over $\mathbb{Q}$. Let $\mathcal{O}$ be the integral closure of $\mathbb{Z}$ in $\mathbb{Q}$.
   (a) Show that $F = \mathbb{Q}[\sqrt{d}]$ for some square-free integer d.
   (b) Determine $\mathcal{O}$ for all d.

(3) A topological space is said to be Noetherian if the closed sets satisfy the descending chain condition. If R is Noetherian, then $\mathrm{Spec}(R)$ is a Noetherian topological space. Is the converse true?

(4) Let R be an integral domain and F be its quotient field. For every maximal ideal $\mathfrak{m}$, one can consider $R_{\mathfrak{m}}$ as a subring of F. Show that $R = \cap_{\mathfrak{m}:\text{ maximal}} R_{\mathfrak{m}}$.

(5) Let R_1, R_2 be integral domains with quotient fields F_1, F_2 respectively. If $f : R_1 \to R_2$ is an isomorphism, then f extends to an isomorphism $F_1 \to F_2$.